DOI:10.3969/J.1SSN.100-5137.2016.03.011

Stationary solution of the compressible magnetohydrodynamic equation and its stability with respect to initial disturbance

WU Renchao

(College of Mathematics and Sciences, Shanghai Normal University, Shanghai 200234)

Abstract: In this paper, we consider three dimensional compressible viscous magnetohydrodynamic equations (MHD) with external potential force. We first derive the corresponding non-constant stationary solutions. Then we show global well-posedness of the initial value problem for the three dimensional compressible viscous magnetohydrodynamic equations, provided that prescribed initial data is close to the stationary solution.

Key words: Magnetohydrodynamic equations; stationary solutions; smooth solutions; energy estimate

CLC number: O29 **Document code:** A **Article ID:** 1000-5137(2016)03-0355-23

2010 MSC: 00A69

In this paper, we are interested in the initial value problem of the compressible viscous magnetohydrodynamic equations:

$$\begin{cases} \rho_t + \nabla\cdot(\rho v) = G(x) \\ v_t + (v\cdot\nabla)v = \frac{\mu}{\rho}\Delta v + \dfrac{\lambda}{\rho}\nabla(\nabla\cdot v) + \dfrac{1}{\rho}(B\cdot\nabla)B - \dfrac{1}{2\rho}\nabla(|B|^2) - \dfrac{\nabla P(\rho)}{\rho} + F(x) \\ B_t + (v\cdot\nabla)B + (\nabla\cdot v)B - (B\cdot\nabla)v = \nu\Delta B + H(x), \nabla\cdot B = 0 \end{cases}, \tag{1}$$

with initial data

$$(\rho, v, B)(t,x)|_{t=0} = (\rho_0, v_0, B_0) \to (\bar{\rho}, 0, 0) \quad \text{as } |x| \to +\infty, \tag{2}$$

where $t \geqslant 0, x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\rho = \rho(t,x) > 0, v = (v^1, v^2, v^3)$ and $B = (B^1, B^2, B^3)$ denote the density, velocity and magnetic field respectively, which are unknown; P is the pressure; μ and λ are the viscosity coefficients which satisfy the condition: $\mu > 0, \lambda + 2\mu/3 \geqslant 0$, ν is the coefficient of the magnetic field conduction; $G(x), F(x) = (F^1(x), F^2(x), F^3(x))$ and $H(x) = (H^1(x), H^2(x), H^3(x))$ are the given external force, mass source and magnetic source. The isentropic compressible viscous magnetohydrodynamic (MHD) equations model the dynamics of compressible quasi-neutrally ionized fluids under the influence of electromagnetic fields and cover a very wide applications of physical objects from liquid metals to cosmic plasmas, see [1-4].

Received date: 2014-10-01

Corresponding author: WU Renchao, E-mail:wurenchao1011@ 126.com

Due to their physical importance, mathematical complexity and wide range of applications, many results concerning the existence and uniqueness of (weak, strong or smooth) solutions for the compressible magnetohydrodynamic equations can be found in [5-16] and the references cited therein. However, we note that all above results are for the compressible magnetohydrodynamic equations without any external force. In this paper, we mainly discuss global existence of the compressible magnetohydrodynamic equation with potential force. Moreover, we also mention that there are a lot of references about the low-mach-limit and inviscid limit for the compressible MHD equation, such as [17-22] and the references therein.

Notations In the rest of this paper, we use the standard notation in vector analysis. For a scalar function u, vectors functions $v=(v^1,v^2,v^3), w=(w^1,w^2,w^3)$ and matrix function $f=(f^{ij})_{1\leqslant i,j\leqslant 3}$, we denote

$$\Delta u=\sum_{i=1}^3\frac{\partial^2 u}{\partial x_i^2},\quad \Delta v=(\Delta v^1,\Delta v^2,\Delta v^3),\quad (v\cdot\nabla)u=\sum_{i=1}^3 v^i\frac{\partial u}{\partial x_i},$$

$$(v\cdot\nabla)w=((v\cdot\nabla)w^1,(v\cdot\nabla)w^2,(v\cdot\nabla)w^3),$$

$$\nabla^k u=(\partial_x^\alpha u\big||\alpha|=k),\quad \nabla^k v=(\partial_x^\alpha v^i\big||\alpha|=k,i=1,2,3),$$

$$\nabla^i u=\frac{\partial u}{\partial x_i},\quad \nabla\cdot v=\sum_{i=1}^3\frac{\partial v^i}{\partial x_i},\quad \nabla\cdot f=\Big(\sum_{j=1}^3\frac{\partial f^{1j}}{\partial x_j},\sum_{j=1}^3\frac{\partial f^{2j}}{\partial x_j},\sum_{j=1}^3\frac{\partial f^{3j}}{\partial x_j}\Big).$$

Here $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is the multi-index, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $\partial_x^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha^1}\partial x_2^{\alpha^2}\partial x_3^{\alpha^3}$.

Next, we introduce some function spaces that will be used later. For scalars functions u_1,u_2 and vectors function $v=(v^1,...,v^n), w=(w^1,...,w^n)$, we put

$$\|u_1\|_{L^p}=\Big(\int_{\mathbb{R}^3}|u_1(x)|^p\mathrm{d}x\Big)^{\frac{1}{p}},\quad \|v\|_{L^p}=\Big(\sum_{i=1}^3\|v^i\|_{L^p}^p\Big)^{\frac{1}{p}}\quad (1\leqslant p<\infty),$$

$$\|u_1\|_{L^\infty}=\sup_{\mathbb{R}^3}|u_1(x)|,\quad \|v\|_{L^\infty}=\max_{1\leqslant i\leqslant n}\|v^i(x)\|_{L^\infty},\quad (u_1,u_2)=\int_{\mathbb{R}^3}u_1u_2\mathrm{d}x,$$

$$(v,\omega)=\sum_{i=1}^n\langle v^i,\omega^i\rangle,\quad \|v\|_k=\Big(\sum_{0\leqslant m\leqslant k}\|\nabla v^m\|^2\Big)^{\frac{1}{2}},\quad \|\cdot\|=\|\cdot\|_{L_2}.$$

Let $L^p(\mathbb{R}^3)$ denote the usual $L^p(\mathbb{R}^3)$ space, and

$$H^k(\mathbb{R}^3)=\{u\in L^1_{\mathrm{loc}}\big|\|u\|_k<\infty\},\quad \widehat{H}^k(\mathbb{R}^3)=\{u\in L^1_{\mathrm{loc}}\big|\nabla u\in H^{k-1}\},$$

where u is either a vector or a scalar.

Definition 1 For $\varepsilon>0$, we introduce

$$I_\varepsilon^k=\{\sigma\in H^k\big|\|\sigma\|_{I^k}<\varepsilon\},\quad J_\varepsilon^k=\{v\in H^k\big|\|v\|_{J^k}<\varepsilon\},$$

where

$$\begin{aligned}\|\sigma\|_{I^k}= &\quad \|\sigma\|_{L^6}+\Big\|\frac{\sigma}{|x|}\Big\|+\sum_{m=1}^k\|(1+|x|)^m\nabla^m\sigma\|+\|(1+|x|)^2\sigma\|_{L^\infty},\\ \|v\|_{J^k}= &\quad \|v\|_{L^6}+\Big\|\frac{v}{|x|}\Big\|+\sum_{m=1}^k\|(1+|x|)^{m-1}\nabla^m v\|+\sum_{m=0}^1\|(1+|x|)^{m+1}\nabla^m v\|_{L^\infty},\end{aligned}$$

In this paper, we consider the case where the mass source G, the external force F and the magnetic source H are given by the following form

$$(G, F, H) = \nabla \cdot (G_1, F_1, H_1) + (G_2, F_2, H_2),$$

where $G_1 = (G_1^i(x))_{1\leqslant i\leqslant 3}, G_2 = G_2(x), \quad F_1 = (F_1^{ij}(x))_{1\leqslant i,j\leqslant 3}, F_2 = (F_2^i(x))_{1\leqslant i\leqslant 3}, \quad H_1 = (H_1^{ij}(x))_{1\leqslant i,j\leqslant 3}$ and $H_2 = (H_2^i(x))_{1\leqslant i\leqslant 3}$.

Now we state the main result of this paper. First, we consider the stationary problem corresponding to the initial problem (1)-(2):

$$\begin{cases} \nabla\cdot(\rho v) = G(x) \\ (v\cdot\nabla)v = \dfrac{\mu}{\rho}\Delta v + \dfrac{\lambda}{\rho}\nabla(\nabla\cdot v) + \dfrac{1}{\rho}(B\cdot\nabla)B - \dfrac{1}{2\rho}\nabla(|B|^2) - \dfrac{\nabla P(\rho)}{\rho} + F(x) \\ (v\cdot\nabla)B + (\nabla\cdot v)B - (B\cdot\nabla)v = \nu\Delta B + H(x), \nabla\cdot B = 0 \end{cases}. \tag{3}$$

The first theorem concerns the existence of stationary solution of (3) and its weighted-L^2, L^∞ estimates.

Theorem 1 Let $\overline{\rho}$ be any positive constant, and $P(\cdot)$ be smooth (at least C^2) in a neighborhood of $\overline{\rho}$. Then, there exist small constants $c_0 > 0$ and $\varepsilon_0 > 0$ depending on $\overline{\rho}$, such that if $(G, F, H) \in H^4 \times H^3 \times H^3$ and satisfies the estimate:

$$\sum_{m=0}^{3} \|(1+|x|)^{m+1}(\nabla^m F, \nabla^m H)\| + \|(1+|x|)^3 (F, H)\|_{L^\infty} + \|(1+|x|)^2 (F_1, H_1)\|_{L^\infty} +$$
$$\|(F_2, H_2)\|_{L^1} + \|(1+|x|)G\| + \sum_{m=0}^{4} \|(1+|x|)^m \nabla^m G\| +$$
$$\sum_{m=0}^{1} \|(1+|x|)^{m+2}\nabla^m G\|_{L^\infty} + \|(1+|x|)^{-1} G\|_{L^1} \leqslant c_0\varepsilon,$$

for some $\varepsilon \leqslant \varepsilon_0$, then (3) admits a solution of the form $(\rho, v, B) = (\overline{\rho} + \sigma, v, B)$ satisfying $\|\sigma\|_{I^4} + \|(v, B)\|_{J^5} \leqslant \varepsilon$ and $\|(1+|x|)^3 V_1\|_{L^\infty} + \|(1+|x|)^{-1} V_2\|_{L^1} \leqslant \varepsilon$, where $\nabla\cdot v = \nabla\cdot V_1 + V_2$. Furthermore the solution is unique in the following sense: if there is another solution $(\overline{\rho} + \sigma_1, v_1, B_1)$ satisfying (3) with the same (G, F, H) and $\|\sigma_1\|_{I^4} + \|(v_1, B_1)\|_{J^5} \leqslant \varepsilon$, then $(\sigma_1, v_1, B_1) = (\sigma, v, B)$.

Next, we consider the stability of the stationary solution of (3) with respect to the initial disturbance. Denote by (ρ^*, v^*, B^*) the stationary solution obtained in Theorem 1. Then the stability of (ρ^*, v^*, B^*) means the solvability of the non-stationary problem (1) with initial data (2). First we introduce a class of functions which solutions of (1)-(2) belong to.

We state the stability theorem.

Theorem 2 There exist constants $C > 0$ and $\delta > 0$ such that if $\|(\rho_0 - \rho^*, v_0 - v^*, B_0 - B^*)\|_3 \leqslant \delta$, then (1) admits a unique solution $(\rho, v, B) = (\rho^* + \varphi, v^* + \psi, B^* + \omega)$ globally in time, where $\varphi(t,x) \in C^0(0,\infty; H^3)\bigcap C^1(0,\infty; H^2), \psi(t,x), \omega(t,x) \in C^0(0,\infty; H^3)\bigcap C^1(0,\infty; H^1), \nabla\varphi, \psi_t, \omega_t \in L^2(0,\infty; H^2)$ and $\nabla\psi,\ \nabla\omega \in L^2(0,\infty; H^3)$. Moreover, the solution (φ, ψ, ω) satisfies the following estimate:

$$\|(\varphi,\psi,\omega)(t)\|_3^2 + \int_0^t \|\nabla\varphi(s)\|_2^2 + \|(\nabla\psi, \nabla\omega)(s)\|_3^2 + \|(\psi_t, \omega_t)(s)\|_2^2 \mathrm{d}s \leqslant C\|(\varphi,\psi,\omega)(0)\|_3^2,$$

for any $t \geqslant 0$.

Remark 1 Under the conditions of Theorem 2, we shall say that the stationary solution (ρ^*, v^*, B^*) of (3) is stable in the H^3-framework with respect to small initial disturbance.

1 Stationary problem

In this section, we study the stationary problem (3). Take any constant $\bar{\rho} > 0$. Substituting $\rho = \bar{\rho} + \sigma$ into (3) and puting $\gamma = P'(\bar{\rho})$, (3) is reduced to the equation:

$$\begin{cases} \nabla \cdot v + (\dfrac{v}{\bar{\rho}+\sigma} \cdot \nabla)\sigma = \dfrac{G}{\bar{\rho}+\sigma} \\ -\mu\Delta v - \lambda\nabla(\nabla \cdot v) + \gamma\nabla\sigma = -(\bar{\rho}+\sigma)(v\cdot\nabla)v - \\ \quad [P'(\bar{\rho}+\sigma) - P'(\bar{\rho})]\nabla\sigma + (\bar{\rho}+\sigma)F + (B\cdot\nabla)B - \dfrac{1}{2}\nabla(|B|^2) \\ -\nu\Delta B = -(v\cdot\nabla)B - (\nabla\cdot v)B + (B\cdot\nabla)v + H \end{cases} . \tag{4}$$

Our goal in this section is to prove Theorem 1 by application of weighted-L^2 method to the linearized problem for (4).

1.1 Linearization of the problem

In this subsection, let $k = 3$ or $k = 4$, and consider the linearized equation of (4):

$$\begin{cases} \nabla \cdot v + (a\cdot\nabla)\sigma = g \\ -\mu\Delta v - \lambda\nabla(\nabla\cdot v) + \gamma\nabla\sigma = f \\ -\nu\Delta B = h \end{cases} , \tag{5}$$

where $a = (a^1(x), a^2(x), a^3(x)), b_1, b_2, c$, and $(g, h, f) \in H^k \times H^{k-1} \times H^{k-1}$ are given as follows:

$$f = -(b_1\cdot\nabla)b_2 + \tilde{f}, \quad h = -(b_1\cdot\nabla)c + \tilde{h}.$$

Throughout this section, we assume that

$$a \in \widehat{H}^4, \quad \|(1+|x|)a\|_{L^\infty} + \sum_{m=1}^{4}\|(1+|x|)^{m-1}\nabla^m a\| \leqslant \delta, \quad b_1, b_2, c \in J_\delta^{k+1}, \tag{6}$$

$$\sum_{m=0}^{k-1}\|(1+|x|)^{m+1}(\nabla^m\tilde{f}, \nabla^m\tilde{h})\| + \|(1+|x|)g\| + \sum_{m=1}^{k}\|(1+|x|)^m\nabla^m g\| \leqslant \infty. \tag{7}$$

We note that the system of equations $(5)_1 - (5)_2$, which are independent of B, has been well studied by Shibata and Tanaka in [23], while $(5)_3$ is easy to be deal with. Thus by repeating the argument in [23], we get the following result for the problem (5).

Theorem 3 There exists $\delta_0 = \delta_0(\mu, \lambda) > 0$ such that if δ in (6) satisfies $\delta \leqslant \delta_0$, then (5) admits a solution $(\sigma, v, B) \in \widehat{H}^k \times \widehat{H}^{k+1} \times \widehat{H}^{k+1}$ which satisfies the following estimate:

$$\begin{aligned} &\|(\sigma, v, B)\|_{L^6} + \left\|\frac{(\sigma, v, B)}{|x|}\right\| + \sum_{m=1}^{k}\|(1+|x|)^m\nabla^m\sigma\| + \sum_{m=1}^{k+1}\|(1+|x|)^{m-1}(\nabla^m v, \nabla^m B)\| \leqslant \\ &\quad C\Big\{\|b_1\|_{J^{k+1}}\|b_2\|_{J^{k+1}} + \|b_1\|_{J^{k+1}}\|c\|_{J^{k+1}} + \|(1+|x|)g\| + \\ &\quad \sum_{m=1}^{k}\|(1+|x|)^m\nabla^m g\| + \sum_{m=0}^{k-1}\|(1+|x|)^{m+1}(\nabla^m\tilde{f}, \nabla^m\tilde{h})\|\Big\}, \end{aligned} \tag{8}$$

where C is a constant depending only on μ, λ and ν.

1.2 Proof of Theorem 1

In this subsection, we shall construct a solution to (4), by use of the contraction mapping principle. Firstly, we consider the following system of equations:

$$\begin{cases} \nabla\cdot v+\Big(\dfrac{\tilde v}{\bar\rho+\tilde\sigma}\cdot\nabla\Big)\sigma=\dfrac{G}{\bar\rho+\tilde\sigma}, \\ -\mu\Delta v-\lambda\nabla(\nabla\cdot v)+\gamma\nabla\sigma=-(\bar\rho+\tilde\sigma)(\tilde v\cdot\nabla)\tilde v- \\ \quad [P'(\bar\rho+\tilde\sigma)-P'(\bar\rho)]\nabla\tilde\sigma+(\bar\rho+\tilde\sigma)F+(\tilde B\cdot\nabla)\tilde B-\dfrac{1}{2}\nabla(|\tilde B|^2) \\ -\nu\Delta B=-(\tilde v\cdot\nabla)\tilde B-(\nabla\cdot\tilde v)\tilde B+(\tilde B\cdot\nabla)\tilde v+H \end{cases}, \tag{9}$$

where $\|\tilde\sigma\|_{I^4}+\|(\tilde v,\tilde B)\|_{J^5}\leqslant\varepsilon$ with $\|(1+|x|)^3\tilde V_1\|_{L^\infty}+\|(1+|x|)^{-1}\tilde V_2\|_{L^1}\leqslant\varepsilon$.

Also, we put

$$\begin{cases} a=\dfrac{\tilde v}{\bar\rho+\tilde\sigma},\quad b_2=\bar\rho b_1=\bar\rho\tilde v,\quad c=\tilde B,\quad g=\dfrac{G}{\bar\rho+\tilde\sigma}, \\ \tilde f=-\tilde\sigma(\tilde v\cdot\nabla)\tilde v-[P'(\bar\rho+\tilde\sigma)-P'(\bar\rho)]\nabla\tilde\sigma+(\bar\rho+\tilde\sigma)F+(\tilde B\cdot\nabla)\tilde B-\frac{1}{2}\nabla(|\tilde B|^2) \\ \tilde h=-(\nabla\cdot\tilde v)\tilde B+(\tilde B\cdot\nabla)\tilde v+H. \end{cases}. \tag{10}$$

We choose $\varepsilon>0$ sufficiently small such that $\dfrac{\bar\rho}{2}<\tilde\rho<2\bar\rho$ and $\delta\leqslant\delta_0$ as follows from the Sobolev inequality. Also we can check directly that (6)-(7) hold for $k=4$. Moreover, we have

$$\sum_{m=0}^{3}\|(1+|x|)^{m+1}(\nabla^m\tilde f,\nabla^m\tilde h)\|+\|(1+|x|)g\|+\sum_{m=1}^{4}\|(1+|x|)^m\nabla^m g\|\leqslant C\{\varepsilon^2+K_0\},$$

for some constant $C=C(\bar\rho,\mu,\lambda,\nu)$, where K_0 is defined by

$$K_0\equiv\|(1+|x|)G\|+\sum_{m=0}^{3}\|(1+|x|)^{m+1}(\nabla^m F,\nabla^m H)\|+\sum_{m=1}^{4}\|(1+|x|)^m\nabla^m G\|<\infty. \tag{11}$$

Secondly, we introduce of the solution map T for (9):

$$T:(\tilde\sigma,\tilde v,\tilde B)\mapsto(\sigma,v,B),$$

that is $I^4_\varepsilon\times J^5_\varepsilon\times J^5_\varepsilon$ with $\|(1+|x|)^3\tilde V_1\|_{L_\infty}+\|(1+|x|)^{-1}\tilde V_2\|_{L^1}\leqslant\varepsilon\mapsto\widehat H^4\times\widehat H^5\times\widehat H^5$.

Now we want to show that

$$\|\tilde\sigma\|_{I^4}+\|(\tilde v,\tilde B)\|_{J^5}\leqslant\varepsilon \text{ with } \|(1+|x|)^3\tilde V_1\|_{L^\infty}+\|(1+|x|)^{-1}\tilde V_2\|_{L^1}\leqslant\varepsilon$$

implies

$$\|\sigma\|_{I^4}+\|(v,B)\|_{J^5}\leqslant\varepsilon \text{ with } \|(1+|x|)^3 V_1\|_{L^\infty}+\|(1+|x|)^{-1}V_2\|_{L^1}\leqslant\varepsilon.$$

Applying Theorem 3 with $k=4$ for (9), we have the following lemma.

Lemma 1 Suppose $(G,F,H)\in H^4\times H^3\times H^3$ satisfy (11). Then there exists ε_0 such that if $\varepsilon\leqslant\varepsilon_0$, then (9) has a solution $(\sigma,v,B)\in\widehat H^4\times\widehat H^5\times\widehat H^5$, which satisfies the following estimate:

$$\|(\sigma,v,B)\|_{L^6}+\left\|\frac{(\sigma,v,B)}{|x|}\right\|+\sum_{m=1}^{5}\|(1+|x|)^{m-1}(\nabla^m v,\nabla^m B)\|+\sum_{m=1}^{4}\|(1+|x|)^m\nabla^m\sigma\|\leqslant C\{\varepsilon^2+K_0\}, \tag{12}$$

where $C>0$ depends only on $\mu,\lambda,\nu,\bar\rho$.

The following lemma plays an important role when we estimate the L^∞-norm of solution.

Lemma 2 Let $E(x)$ be a scalar function satisfying

$$|\partial_x^\alpha E(x)| \leqslant \frac{C_\alpha}{|x|^{|\alpha|+1}} \quad (|\alpha| = 0, 1, 2).$$

(i) If $\phi(x)$ is a smooth scalar function of the form $\phi = \nabla \cdot \phi_1 + \phi_2$ satisfying

$$L_1(\phi) \equiv \|(1+|x|)^3 \phi\|_{L^\infty} + \|(1+|x|)^2 \phi_1\|_{L^\infty} + \|\phi_2\|_{L^1} < \infty,$$

then for any multi-index α with $|\alpha| = 0, 1$, we have

$$|\partial_x^\alpha (E * \phi)(x)| \leqslant \frac{C_\alpha}{|x|^{|\alpha|+1}} L_1(\phi).$$

(ii) If $\phi(x)$ is a smooth scalar function of the form $\phi = \phi_1 \phi_2$ satisfying

$$L_2(\phi) \equiv \|(1+|x|)^2 \phi\|_{L^\infty} + \|(1+|x|)^3 (\nabla \phi_1) \phi_2\|_{L^\infty} + \|(1+|x|)^3 \phi_1 (\nabla \phi_2)\|_1 < \infty,$$

then for any multi-index α with $|\alpha| = 1, 2$, we have

$$|\partial_x^\alpha (E * \phi)(x)| \leqslant \frac{C_\alpha}{|x|^{|\alpha|}} L_2(\phi).$$

Here, C_α denotes a constant depending only on α.

This lemma can be proved by partitioning $\mathbb{R}^3$ and estimating the integral in each domain of $\mathbb{R}^3$ respectively, cf. [24].

Now, with Helmholtz decomposition and Fourier transform, we shall estimate the L^∞-norm of the solution to (9).

Lemma 3 Let (G, H, F) satisfy the following estimate (for K_0 defined by (11)):

$$K \equiv K_0 + \|(1+|x|)^3 (F, H)\|_{L^\infty} + \|(1+|x|)^2 (F_1, H_1)\|_{L^\infty} + \|(F_2, H_2)\|_{L^1} + \sum_{m=0}^{1} \|(1+|x|)^{m+2} \nabla^m G\|_{L^\infty} < \infty.$$

Then, if $(\sigma, v, B) \in \widehat{H}^4 \times \widehat{H}^5 \times \widehat{H}^5$ is a solution to (9), which satisfies (12) and $\|\tilde{\sigma}\|_{I^4} + \|(\tilde{v}, \tilde{B})\|_{J^5} \leqslant \varepsilon$ with $\|(1+|x|)^3 \tilde{V}_1\|_{L^\infty} + \|(1+|x|)^{-1} \tilde{V}_2\|_{L^1} \leqslant \varepsilon$, then (σ, v, B) satisfies the following estimate:

$$\|(1+|x|)^2 \sigma\|_{L^\infty} + \sum_{m=0}^{1} \|(1+|x|)^{m+1} (\nabla^m v, \nabla^m B)\|_{L^\infty} \leqslant C\{\varepsilon^2 + K\}, \tag{13}$$

where the constant $C > 0$ depends only μ, λ, ν and $\bar{\rho}$.

Proof In view of Helmholtz decomposition, v is written in the form:

$$v = w + \nabla p \quad (w \in \dot{L}^6, \nabla p \in M^6). \tag{14}$$

Here and hereafter

$$M^6 = \{\nabla p \big| p \in L^6_{\text{loc}}, \nabla p \in L^6\}, \quad \dot{L}^6 = \overline{\{w \in C_0^\infty \big| \nabla \cdot w = 0\}}^{\|\cdot\|_{L^6}},$$

where $\bar{\cdot}^{\|\cdot\|_{L^6}}$ means the completion of $\cdot$ with respect to the L^6-norm. Substituting (14) into (9), we have

$$\begin{cases} \Delta p + (\frac{\tilde{v}}{\bar{\rho}+\tilde{\sigma}}\cdot\nabla)\sigma = \frac{G}{\bar{\rho}+\tilde{\sigma}} \\ -\mu\Delta w + \nabla\Phi = f \\ \Phi = \gamma\sigma - (\lambda+\mu)\Delta p \\ -\nu\Delta B = h \end{cases}, \tag{15}$$

where f, h is defined by

$$f = -\bar{\rho}(\tilde{v}\cdot\nabla)\tilde{v} + \tilde{f}, \quad h = -(\tilde{v}\cdot\nabla)\tilde{B} + \tilde{h} \quad (\tilde{f}, \tilde{h} \text{ is what we put at (10)}),$$

so we get the representation w, p, Φ, B

$$\begin{cases} w = \sum_{i=1}^{3} E^{ij} * f^i(x) \\ p = E_0 * R(x) \\ \Phi = \sum_{i=1}^{3} \frac{\partial E_0}{\partial x_i} * f^i(x) \\ B = E_0 * \xi \end{cases}, \tag{16}$$

and

$$\begin{cases} E^{ij} = \frac{1}{8\pi\mu}(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}), \quad E_0 = -\frac{1}{4\pi}|x|^{-1} \\ f^i = -\bar{\rho}(\tilde{v}\cdot\nabla)\tilde{v}^i + \tilde{f}^i \\ R = -(\frac{\tilde{v}}{\bar{\rho}+\tilde{\sigma}}\cdot\nabla)\sigma + \frac{G}{\bar{\rho}+\tilde{\sigma}} \\ \xi = -\frac{h}{\nu} = \frac{1}{\nu}\{(\tilde{v}\cdot\nabla)\tilde{B} - \tilde{h}\} \end{cases}. \tag{17}$$

We shall apply Lemma 2 (i) to estimate w, Φ and B. Therefore, in order to estimate $(16)_1,\ (16)_3$ and $(16)_4$, we need to take a look at f, h. By $(\tilde{\sigma}, \tilde{v}, \tilde{B})$ there exist $\tilde{V}_1 = (\tilde{V}_1^i)_{1\leqslant i\leqslant 3}$ and $\tilde{V}_2$ such that

$$\nabla\cdot\tilde{v} = \nabla\cdot\tilde{V}_1 + \tilde{V}_2, \quad \|(1+|x|)^3\tilde{V}_1\|_{L^\infty} + \|(1+|x|)^{-1}\tilde{V}_2\|_{L^1} \leqslant \varepsilon, \tag{18}$$

and so we can calculate

$$f^i = -(\bar{\rho}+\tilde{\sigma})(\tilde{v}\cdot\nabla)\tilde{v}^i + (\tilde{B}\cdot\nabla)\tilde{B}^i - \frac{1}{2}\nabla(|\tilde{B}^i|^2) - \{P'(\bar{\rho}+\tilde{\sigma}) - P'(\bar{\rho})\}\frac{\partial\tilde{\sigma}}{\partial x_i} + (\bar{\rho}+\tilde{\sigma})F^i =$$

$$\big[\bar{\rho}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\{-\tilde{v}^i\tilde{v}^j + \tilde{v}^i\tilde{V}_1^j\} + \nabla\cdot\{(\bar{\rho}+\tilde{\sigma})F_1^i\}\big] + [-\bar{\rho}(\tilde{V}_1\cdot\nabla)\tilde{v}^i - \bar{\rho}\tilde{V}_2\tilde{v}^i - \tilde{\sigma}(\tilde{v}\cdot\nabla)\tilde{v}^i - Q(\tilde{\sigma})\tilde{\sigma}\frac{\partial\tilde{\sigma}}{\partial x_i} -$$

$$\nabla\tilde{\sigma}\cdot F_1^i + (\bar{\rho}+\tilde{\sigma})F_2^i + (\tilde{B}\cdot\nabla)\tilde{B}^i - \frac{1}{2}\nabla(|\tilde{B}^i|^2)\big] \equiv \nabla\cdot f_1^i + f_2^i,$$

and

$$\xi = \frac{1}{\nu}\{-(\tilde{v}\cdot\nabla)\tilde{B} - (\nabla\cdot\tilde{v})\tilde{B} + (\tilde{B}\cdot\nabla)\tilde{v} + H\} =$$

$$\big[\sum_{i=1}^{3} -\frac{1}{\nu}\frac{\partial}{\partial x_i}(\tilde{v}^i\tilde{B}) + \nabla\cdot H_1\big] + \big[(\tilde{B}\cdot\nabla)\tilde{v} + H_2\big] =$$

$$\nabla\cdot\xi_1 + \xi_2,$$

where

$$Q(\sigma)=\int_0^1 P''(\bar{\rho}+\theta\sigma)\mathrm{d}\theta.$$

Since $\|\tilde{\sigma}\|_{I^4}+\|(\tilde{v},\tilde{B})\|_{J^5}\leqslant\varepsilon$ and $\|(1+|x|)^3\tilde{V}_1\|_{L^\infty}+\|(1+|x|)^{-1}\tilde{V}_2\|_{L^1}\leqslant\varepsilon$, by Sobolev inequality we obtain from (18) that

$$\|(1+|x|)^3(f^i,\xi)\|_{L^\infty}+\|(1+|x|)^2(f_1^i,\xi_1)\|_{L^\infty}+\|(f_2^i,\xi_2)\|_{L^1}\leqslant C\{\varepsilon^2+K_1\},$$

where K_1 is defined by

$$K_1=\|(1+|x|)^3(F,H)\|_{L^\infty}+\|(1+|x|)^2(F_1,H_1)\|_{L^\infty}+\|(F_2,H_2)\|_{L^1},$$

and $C>0$ is a constant depending only on $\bar{\rho}$. Thus, applying Lemma 2 (i), we obtain

$$|x|^2|\varPhi(x)|+\sum_{m=0}^{1}|x|^{m+1}|(\nabla^m w(x),\nabla^m B(x))|\leqslant CK_1. \tag{19}$$

As for p, we have from $(16)_2$ that

$$p=E_0*\Big\{-\sum_{i=1}^{3}\frac{\tilde{v}^i}{\bar{\rho}+\tilde{\sigma}}\frac{\partial\sigma}{\partial x_i}+\frac{G}{\bar{\rho}+\tilde{\sigma}}\Big\}\equiv -E_0*\sum_{i=1}^{3}q_1^iq_2^i+E_0*r. \tag{20}$$

Since $\|\tilde{\sigma}\|_{I^4}+\|(\tilde{v},\tilde{B})\|_{J^5}\leqslant\varepsilon$ and $\|(1+|x|)^3\tilde{V}_1\|_{L^\infty}+\|(1+|x|)^{-1}\tilde{V}_2\|_{L^1}\leqslant\varepsilon$, by Sobolev inequality we obtain from (12) that

$$\|(1+|x|)^2q_1^iq_2^i\|_{L^\infty}+\|(1+|x|)^3(\nabla q_1^i)q_2^i\|_{L^\infty}+\|(1+|x|)^3q_1^i\nabla q_2^i\|_{L^1}\leqslant C\{\varepsilon^2+K_0\},$$

$$\sum_{m=0}^{1}\|(1+|x|)^{m+2}\nabla^m r\|_{L^\infty}\leqslant C\sum_{m=0}^{1}\|(1+|x|)^{m+2}\nabla^m G\|_{L^\infty}\equiv K_2,$$

where the constant $C>0$ depends only on $\bar{\rho}$. Applying Lemma 2 (ii) to each term of (20), we also have

$$\sum_{m=1}^{2}|x|^m|\nabla^m p(x)|\leqslant C\{\varepsilon^2+K_0+K_2\}. \tag{21}$$

Now we consider the case $|x|>1$. Returning to (14) and combining (19) and (21), we obtain

$$\sum_{m=0}^{1}(1+|x|)^{m+1}|\nabla^m v(x)|\leqslant C\{\varepsilon^2+K_0+K_1+K_2\}. \tag{22}$$

Besides, by $(15)_3$ we have

$$\sigma=\gamma^{-1}\{(\mu+\lambda)\Delta p+\varPhi\}.$$

Combining (19) and (21), we have

$$(1+|x|)^2|\sigma(x)|\leqslant C\{\varepsilon^2+K_0+K_1+K_2\}. \tag{23}$$

Moreover, from (19), we get

$$\sum_{m=0}^{1}(1+|x|)^{m+1}|\nabla^m B(x)|\leqslant C\{\varepsilon^2+K_0+K_1+K_2\}. \tag{24}$$

Finally we consider the case $|x| < 1$. Sobolev inequality and Hardy inequality imply that

$$(1+|x|)^2|\sigma(x)| + \sum_{m=0}^{1}(1+|x|)^{m+1}\big|\big(\nabla^m v(x), \nabla^m B(x)\big)\big| \leqslant$$
$$8\left\|\frac{\sigma}{1+|x|}\right\|_2 + 4\|(\nabla v, \nabla B)\|_2 + 4\left\|\frac{(v,B)}{1+|x|}\right\|_2 \leqslant C\big(\|\nabla\sigma\|_1 + \|(\nabla v,\nabla B)\|_2\big) \leqslant C\{\varepsilon^2 + K_0\}. \tag{25}$$

This completes the proof of Lemma 3.

The main purpose of the following proposition is to prove that $\|\sigma\|_{I^4}+\|(v,B)\|_{J^5} \leqslant \varepsilon$ with $\|(1+|x|)^3 V_1\|_{L^\infty}+\|(1+|x|)^{-1}V_2\|_{L^1} \leqslant \varepsilon$.

Proposition 1 There exist c_0 and $\varepsilon > 0$ such that (9) admits a solution $(\sigma, v, B) = T(\tilde\sigma, \tilde v, \tilde B)$ satisfying $\|\sigma\|_{I^4} + \|(v,B)\|_{J^5} \leqslant \varepsilon$ with $\|(1+|x|)^3V_1\|_{L^\infty} + \|(1+|x|)^{-1}V_2\|_{L^1} \leqslant \varepsilon$ for all $(G,F,H) \in H^4\times H^3\times H^3$ satisfies

$$K + \|(1+|x|)^{-1}G\|_{L^1} \leqslant c_0\varepsilon, \tag{26}$$

where K is defined in Lemma 3.

Proof By Lemmas 1 and 3, there exists a solution $(\sigma, v, B) \in \widehat{H}^4\times\widehat{H}^5\times\widehat{H}^5$ for (9), satisfying

$$\|\sigma\|_{I^4} + \|(v,B)\|_{J^5} \leqslant C\{\varepsilon^2 + K\} \leqslant C\{\varepsilon^2 + c_0\varepsilon\},$$

where the constant $C > 0$ depends only on μ, λ, ν and $\bar\rho$. Thus if we take $c_0 \leqslant \frac{1}{2C}$ and $\varepsilon > 0$ sufficiently small, it follows that $\|\sigma\|_{I^4} + \|(v,B)\|_{J^5} \leqslant \varepsilon$. Now we define V_1 and V_2 by

$$V_1 = -\frac{\tilde v}{\bar\rho+\tilde\sigma}\sigma, \quad V_2 = \left(\nabla\cdot\frac{\tilde v}{\bar\rho+\tilde\sigma}\right)\sigma + \frac{G}{\bar\rho+\tilde\sigma},$$

and $(9)_1$ gives

$$\nabla\cdot v = \nabla\cdot V_1 + V_2.$$

Moreover, by $\|(1+|x|)^3\tilde V_1\|_{L^\infty} + \|(1+|x|)^{-1}\tilde V_2\|_{L^1} \leqslant \varepsilon$ and (13), we use Sobolev inequality to get

$$\|(1+|x|)^3V_1\|_{L^\infty} + \|(1+|x|)^{-1}V_2\|_{L^1} \leqslant C\{\varepsilon^2 + K + \|(1+|x|)^{-1}G\|_{L^1}\} \leqslant$$
$$C\{\varepsilon^2 + c_0\varepsilon\} \leqslant C\varepsilon^2 + \frac{\varepsilon}{2} \leqslant \varepsilon,$$

if $c_0 \leqslant \frac{1}{2C}$ and $\varepsilon > 0$ are sufficiently small. This completes the proof of Proposition 1.

1.3 Contraction of the solution map T

Finally, we shall show that the solution map T for (9) is contract. We suppose that

$$\|\tilde\sigma^j\|_{I^4} + \|(\tilde v^j, \tilde B^j)\|_{J^5} \leqslant \varepsilon \text{ with } \|(1+|x|)^3\tilde V_1^j\|_{L^\infty} + \|(1+|x|)^{-1}\tilde V_2^j\|_{L^1} \leqslant \varepsilon,$$

and that $(\sigma^j, v^j, B^j) = T(\tilde\sigma^j, \tilde v^j, \tilde B^j)$ satisfies

$$\|\sigma^j\|_{I^4} + \|(v^j, B^j)\|_{J^5} \leqslant \varepsilon \text{ with } \|(1+|x|)^3V_1^j\|_{L^\infty} + \|(1+|x|)^{-1}V_2^j\|_{L^1} \leqslant \varepsilon,$$

for $j = 1, 2$. Then it follows immediately from (9) that

$$\begin{cases} \nabla\cdot(v^1-v^2) + \Big(\dfrac{\tilde v^1}{\bar\rho+\tilde\sigma^1}\cdot\nabla\Big)(\sigma^1-\sigma^2) = g \\ -\mu\Delta(v^1-v^2) - \lambda\nabla(\nabla\cdot(v^1-v^2)) + \gamma\nabla(\sigma^1-\sigma^2) = \\ \quad -\bar\rho(\tilde v^1\cdot\nabla)\tilde v^1 + \bar\rho(\tilde v^2\cdot\nabla)\tilde v^2 + \tilde f \\ -\nu\Delta(B^1-B^2) = -(\tilde v^1\cdot\nabla)\tilde B^1 + (\tilde v^2\cdot\nabla)\tilde B^2 + \tilde h \end{cases}, \tag{27}$$

where $(g,\tilde f,\tilde h)\in H^3\times H^3\times H^3$ is defined by

$$g=-\Big(\frac{\tilde v^1}{\bar\rho+\tilde\sigma^1}-\frac{\tilde v^2}{\bar\rho+\tilde\sigma^2}\Big)\cdot\nabla\sigma^2+\Big(\frac{G}{\bar\rho+\tilde\sigma^1}-\frac{G}{\bar\rho+\tilde\sigma^2}\Big),$$
$$\tilde f=-\tilde\sigma^1(\tilde v^1\cdot\nabla)\tilde v^1+\tilde\sigma^2(\tilde v^2\cdot\nabla)\tilde v^2-[P'(\bar\rho+\tilde\sigma^1)-P'(\bar\rho)]\nabla\tilde\sigma^1+$$
$$[P'(\bar\rho+\tilde\sigma^2)-P'(\bar\rho)]\nabla\tilde\sigma^2+(\tilde\sigma^1-\tilde\sigma^2)F+(\tilde B^1\cdot\nabla)\tilde B^1-$$
$$(\tilde B^2\cdot\nabla)\tilde B^2-\tfrac12\nabla(|\tilde B^1|^2)+\tfrac12\nabla(|\tilde B^2|^2),$$
$$\tilde h=-(\nabla\cdot\tilde v^1)\tilde B^1+(\nabla\cdot\tilde v^2)\tilde B^2+(\tilde B^1\cdot\nabla)\tilde v^1-(\tilde B^2\cdot\nabla)\tilde v^2.$$

Since

$$\sum_{m=0}^{3}\|(1+|x|)^{m+1}\nabla^m(\tilde f,\tilde h)\|+\|(1+|x|)g\|+\sum_{m=1}^{3}\|(1+|x|)^m\nabla^m g\|\leqslant$$
$$C\{\varepsilon+K_0\}\big\{\|\tilde\sigma^1-\tilde\sigma^2\|_{I^3}+\|(\tilde v^1-\tilde v^2,\tilde B^1-\tilde B^2)\|_{J^4}\big\},$$

from Sobolev inequality for K_0 defined in (11), by application of Theorem 3 with $k=3$ to (27), we obtain

$$\|(\sigma^1-\sigma^2,v^1-v^2,B^1-B^2)\|_{L^6}+\left\|\frac{(\sigma^1-\sigma^2,v^1-v^2,B^1-B^2)}{|x|}\right\|+$$
$$\sum_{m=1}^{3}\|(1+|x|)^m\nabla^m(\sigma^1-\sigma^2)\|+\sum_{m=1}^{4}\|(1+|x|)^{m-1}\big(\nabla^m(v^1-v^2),\nabla^m(B^1-B^2)\big)\|\leqslant$$
$$C\{\varepsilon+K_0\}\big\{\|\tilde\sigma^1-\tilde\sigma^2\|_{I^3}+\|(\tilde v^1-\tilde v^2,\tilde B^1-\tilde B^2)\|_{J^4}\big\}.\tag{28}$$

Next, we decompose (27) as in the proof of Lemma 3. Putting $v^1-v^2=w+\nabla p$, we have

$$\begin{cases}\Delta p+\Big(\dfrac{\tilde v^1}{\bar\rho+\tilde\sigma^1}\cdot\nabla\Big)(\sigma^1-\sigma^2)=g\\ -\mu\Delta w+\nabla\Phi=-\bar\rho(\tilde v^1\cdot\nabla)\tilde v^1+\bar\rho(\tilde v^2\cdot\nabla)\tilde v^2+\tilde f=f\\ \Phi=\gamma(\sigma^1-\sigma^2)-(\lambda+\mu)\Delta p\\ -\nu\Delta(B^1-B^2)=-(\tilde v^1\cdot\nabla)\tilde B^1+(\tilde v^2\cdot\nabla)\tilde B^2+\tilde h=h\end{cases}.$$

Similarly, by using the same argument as in the proof of Lemma 3, we have

$$\|(1+|x|)^2(\sigma^1-\sigma^2)\|_{L^\infty}+\sum_{m=0}^{1}\|(1+|x|)^{m+1}\nabla^m(v^1-v^2,B^1-B^2)\|_{L^\infty}\leqslant$$
$$C\{\varepsilon+K\}\big(\|\tilde\sigma^1-\tilde\sigma^2\|_{I^3}+\|(\tilde v^1-\tilde v^2,\tilde B^1-\tilde B^2)\|_{J^4}\big)+$$
$$C\varepsilon\big[\|(1+|x|)^3(\tilde V_1^1-\tilde V_1^2)\|_{L^\infty}+\|(1+|x|)^{-1}(\tilde V_2^1-\tilde V_2^2)\|_{L^1}\big],\tag{29}$$

where $\tilde V_1^j,\tilde V_2^j(j=1,2)$ are functions satisfying

$$\nabla\cdot\tilde v^j=\nabla\cdot\tilde V_1^j+\tilde V_2^j,\quad \|(1+|x|)^3\tilde V_1^j\|_{L^\infty}+\|(1+|x|)^{-1}\tilde V_2^j\|_{L^1}\leqslant\varepsilon.\tag{30}$$

Moreover, if we take $V_1^j,V_2^j(j=1,2)$ as

$$V_1^j=-\frac{\tilde v^j}{\bar\rho+\tilde\sigma^j}\sigma^j,\quad V_2^j=\left(\nabla\cdot\frac{\tilde v^j}{\bar\rho+\tilde\sigma^j}\right)\sigma^j+\frac{G}{\bar\rho+\tilde\sigma^j},\tag{31}$$

then

$$\|(1+|x|)^3(V_1^1-V_1^2)\|_{L^\infty}+\|(1+|x|)^{-1}(V_2^1-V_2^2)\|_{L^1}\leqslant$$
$$C\{\varepsilon+\|(1+|x|)^{-1}G\|_{L_1}\}\big(\|\tilde\sigma^1-\tilde\sigma^2\|_{I^3}+\|(\tilde v^1-\tilde v^2,\tilde B^1-\tilde B^2)\|_{J^4}\big).\tag{32}$$

Combining (28), (29) and (32), we obtain

$$\|\sigma^1-\sigma^2\|_{I^3}+\|(v^1-v^2,B^1-B^2)\|_{J^4}+$$
$$\|(1+|x|)^3(V_1^1-V_1^2)\|_{L_\infty}+\|(1+|x|)^{-1}(V_2^1-V_2^2)\|_{L_1}\leqslant$$
$$C\{\varepsilon+K\}\big(\|\tilde\sigma^1-\tilde\sigma^2\|_{I^3}+\|(\tilde v^1-\tilde v^2,\tilde B^1-\tilde B^2)\|_{J^4}\big)+$$
$$C\varepsilon\big\{\|(1+|x|)^3(\tilde V_1^1-\tilde V_1^2)\|_{L_\infty}+\|(1+|x|)^{-1}(\tilde V_2^1-\tilde V_2^2)\|_{L_1}\big\}.$$

Therefore, we have the following proposition.

Proposition 2 There exist $c_0>0$ and $\varepsilon>0$ such that if $(G,F,H)\in H^4\times H^3\times H^3$ satisfies

$$K\leqslant c_0\varepsilon\quad (K \text{ is defined in Lemma 3}),$$

then for

$$\|\tilde\sigma^j\|_{I^4}+\|(\tilde v^j,\tilde B^j)\|_{J^5}\leqslant\varepsilon \text{ with } \|(1+|x|)^3\tilde V_1^j\|_{L^\infty}+\|(1+|x|)^{-1}\tilde V_2^j\|_{L^1}\leqslant\varepsilon,$$

and $(\sigma^j,v^j,B^j)=T(\tilde\sigma^j,\tilde v^j,\tilde B^j)$ $(j=1,2)$, we have the following estimate:

$$\|\sigma^1-\sigma^2\|_{I^3}+\|(v^1-v^2,B^1-B^2)\|_{J^4}+$$
$$\|(1+|x|)^3(V_1^1-V_1^2)\|_{L^\infty}+\|(1+|x|)^{-1}(V_2^1-V_2^2)\|_{L^1}\leqslant$$
$$\tfrac{1}{2}\Big\{\big(\|\tilde\sigma^1-\tilde\sigma^2\|_{I^3}+\|(\tilde v^1-\tilde v^2,\tilde B^1-\tilde B^2)\|_{J^4}\big)+$$
$$\|(1+|x|)^3(\tilde V_1^1-\tilde V_1^2)\|_{L^\infty}+\|(1+|x|)^{-1}(\tilde V_2^1-\tilde V_2^2)\|_{L^1}\Big\},$$

where $(\tilde V_1^j,\tilde V_2^j)$ $(j=1,2)$ satisfy (30), and $\tilde V_1^j,\tilde V_2^j$ is defined by (31).

Hence, by Propositions 1 and 2, the contraction mapping principle implies the existence and uniqueness of solutions to (3). This completes the proof of Theorem 1.

2 Reformulation of the original problem

In this section, we consider stability of the stationary solution with respect to the initial disturbance (ρ_0,v_0,B_0). Let $\bar\rho$ be a positive constant and let (G,F,H) be small in the sense of Theorem 1. We denote the corresponding stationary solution obtained in Theorem 1.1 by (ρ^*,v^*,B^*). Putting

$$\rho(t,x)=\rho^*+\varphi(t,x),\quad v(t,x)=v^*+\psi(t,x),\quad B(t,x)=B^*+\omega(t,x)$$

into (1.1), we have the system of equations for (φ,ψ,ω):

$$\begin{cases}\varphi_t+\nabla\cdot\{(\rho^*+\varphi)\psi\}=-\nabla\cdot(v^*\varphi)\\ \psi_t-\dfrac{1}{\rho^*}\{\mu\Delta\psi+\lambda\nabla(\nabla\cdot\psi)\}+A_1\nabla\varphi=R_1\\ \omega_t-\nu\Delta\omega=R_2\end{cases},\tag{33}$$

where

$$R_1(t) = -(v^*\cdot\nabla)\psi - (\psi\cdot\nabla)(v^*+\psi) + \frac{1}{\rho^*+\varphi}\big((B^*+\omega)\cdot\nabla\big)\omega - \frac{B^*+\omega}{\rho^*+\varphi}\nabla\omega -$$
$$\frac{1}{\rho^*}\{(P'(\rho^*+\varphi)-P'(\rho^*))\nabla\rho^* - (\omega\cdot\nabla)B^* + \omega\cdot\nabla B^*\} -$$
$$\frac{\varphi}{\rho^*(\rho^*+\varphi)}\big\{\mu\Delta(v^*+\psi) + \lambda\nabla(\nabla\cdot(v^*+\psi)) - P'(\rho^*+\varphi)\nabla\rho^* +$$
$$((B^*+\omega)\cdot\nabla)B^* - (B^*+\omega)\cdot\nabla B^*\big\},$$
$$R_2(t) = \big((B^*+\omega)\cdot\nabla\big)\psi - (B^*+\omega)\nabla\cdot\psi - (\nabla\cdot v^*)\omega +$$
$$(\omega\cdot\nabla)v^* - (\psi\cdot\nabla)B^* - \big((v^*+\psi)\cdot\nabla\big)\omega,$$

and

$$A_1 = \frac{P'(\rho^*+\varphi)}{\rho^*+\varphi}.$$

In this section, our goal is to give a proof of Theorem 2. The proof consists of two steps. The first step is to prove local existence.

Lemma 4 If $(\varphi,\psi,\omega)\in H^3\times H^3\times H^3$, then there exists $t>0$ such that the initial value problem (33) with initial data $(\varphi,\psi,\omega)(0)$ admits a unique solution $(\varphi,\psi,\omega)(t)$ with $\varphi(t,x)\in C^0(0,t_0;H^3)\bigcap C^1(0,t_0;H^2)$ and $\psi(t,x),\omega(t,x)\in C^0(0,t_0;H^3)\bigcap C^1(0,t_0;H^1)$. Moreover, (φ,ψ,ω) satisfies

$$\|(\varphi,\psi,\omega)(t)\|_3^2 \leqslant 2\|(\varphi,\psi,\omega)(0)\|_3^2,$$

for any $t\in[0,t_0]$.

The second step is to prove an estimate.

Lemma 5 Let $(\varphi,\psi,\omega)(t)$ be a solution to (33) with $\varphi(t,x)\in C^0(0,t_1;H^3)\bigcap C^1(0,t_1;H^2)$, and $\psi(t,x)$, $\omega(t,x)\in C^0(0,t_1;H^3)\bigcap C^1(0,t_1;H^1)$. Then there exists $\varepsilon>0$ such that if $\varepsilon\leqslant\varepsilon_0$ and $\sup_{0\leqslant t\leqslant t_1}\|(\varphi,\psi,\omega)(t)\|_3$, $\|\rho^*-\bar\rho\|_{I_4}+\|(v^*,B^*)\|_{J_5}\leqslant\varepsilon$, then

$$\|(\varphi,\psi,\omega)(t)\|_3^2 + \int_0^t \|\nabla\varphi(s)\|_2^2 + \|(\nabla\psi,\nabla\omega)(s)\|_3^2 + \|(\psi_t,\omega_t)(t)\|_2^2\mathrm{d}s \leqslant C\|(\varphi,\psi,\omega)(0)\|_3^2, \tag{34}$$

for any $t\in[0,t_1]$, where $C>0$ is a constant depending only μ,λ and ν.

Concerning the local existence, we can apply the Matsumura and Nishida [25] method directly. So we shall devote the following section to the proof of Lemma 5.

Remark 2 In the following lemmas and proofs, small number ε satisfies

$$\|(\varphi,\psi,\omega)(t)\|_3,\quad \|\rho^*-\bar\rho\|_{I^4}+\|(v^*,B^*)\|_{J^5}\leqslant\varepsilon<\frac{\bar\rho}{4},$$

so that $3\bar\rho/4\leqslant\rho^*\leqslant5\bar\rho/4,\ \bar\rho/2\leqslant\rho^*+\varphi(t)\leqslant3\bar\rho/2$ etc.

3 Proof of Lemma 5

3.1 The estimate for $\nabla\psi(t),\nabla\omega(t)$ and their derivatives up to $\nabla^4\psi(t),\nabla^4\omega(t)$

Lemma 6 Let $(\varphi,\psi,\omega)(t)$ be a solution to (33) as in Lemma 5 for any $t\in[0,t_1]$. Then there exist $\varepsilon_0>0,\zeta_0>0$ and $\alpha_0>0$ such that if $\varepsilon\leqslant\varepsilon_0$ and $\|(\varphi,\psi,\omega)(t)\|_3,\|\rho^*-\bar\rho\|_{I^4}+\|(v^*,B^*)\|_{J^5}\leqslant\varepsilon$, the following lower order derivative estimate holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\varphi(t)\|^2+\int_{\mathbb{R}^3}A_2(t)\psi(t)^2\mathrm{d}x+\int_{\mathbb{R}^3}\omega(t)^2\mathrm{d}x\right]+\alpha_0\|(\nabla\psi(t),\nabla\omega(t))\|^2\leqslant C\varepsilon\|\nabla\varphi(t)\|^2, \tag{35}$$

for any ζ with $0\leqslant\zeta\leqslant\zeta_0$, where $C>0$ is a constant depending on μ,λ and ν, and

$$A_2=(\rho^*+\varphi)A_1^{-1}=\frac{(\rho^*+\varphi)^2}{P'(\rho^*+\varphi)}.$$

Proof Using the Friedrichs mollifier, we may assume that $(\varphi,\psi,\omega)(t)\in C^0(0,t_1;H^\infty)\bigcap C^1(0,t_1;H^\infty)$. Multiplying $(33)_1,(33)_2$ and $(33)_3$, by $\varphi(t),A_2\psi(t)$ and $\omega(t)$ respectively, and integrating the final result with respect to x over $\mathbb{R}^3$, we have

$$\begin{gathered}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|^2-\int_{\mathbb{R}^3}(\rho^*+\varphi(t))\psi(t)\nabla\varphi(t)\mathrm{d}x=\int_{\mathbb{R}^3}v^*\varphi(t)\nabla\varphi(t)\mathrm{d}x,\\ \int_{\mathbb{R}^3}A_2(t)\psi_t(t)\psi(t)\mathrm{d}x+\int_{\mathbb{R}^3}(\rho^*+\varphi(t))\psi(t)\nabla\varphi(t)\mathrm{d}x=\\ \int_{\mathbb{R}^3}R_1(t)A_2(t)\psi(t)\mathrm{d}x+\int_{\mathbb{R}^3}\frac{A_2(t)}{\rho^*}\Big\{\mu\Delta\psi(t)+\lambda\nabla(\nabla\cdot\psi(t))\Big\}\psi(t)\mathrm{d}x,\\ \int_{\mathbb{R}^3}\omega_t(t)\omega(t)\mathrm{d}x=\int_{\mathbb{R}^3}R_2(t)\omega(t)\mathrm{d}x+\int_{\mathbb{R}^3}\nu\omega(t)\Delta\omega(t)\mathrm{d}x,\end{gathered}$$

Then, canceling the term of $\int_{\mathbb{R}^3}(\rho^*+\varphi(t))\psi(t)\nabla\varphi(t)\mathrm{d}x$ by adding the above three formulas, writing the first term of the second formula and the third formula as follows:

$$\begin{aligned}&\int_{\mathbb{R}^3}A_2(t)\psi_t(t)\psi(t)\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}A_2(t)\psi(t)^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}A_{2t}\psi(t)^2\mathrm{d}x,\\ &\int_{\mathbb{R}^3}\omega_t(t)\omega(t)\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\omega(t)^2\mathrm{d}x,\end{aligned}$$

and using integration by parts, we have

$$\begin{gathered}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\varphi(t)\|^2+\int_{\mathbb{R}^3}A_2(t)\psi(t)^2\mathrm{d}x+\int_{\mathbb{R}^3}\omega(t)^2\mathrm{d}x\right\}+\frac{\mu A_2(t)}{\rho^*}\|\nabla\psi(t)\|^2+\nu\|\nabla\omega(t)\|^2\leqslant\\ \left|\int_{\mathbb{R}^3}v^*\varphi(t)\nabla\varphi(t)\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}R_1(t)A_2(t)\psi(t)\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}R_2(t)\omega(t)\mathrm{d}x\right|+\\ \frac{1}{2}\left|\int_{\mathbb{R}^3}A_{2t}(t)\psi(t)^2\mathrm{d}x\right|+\left\{\left|\mu\int_{\mathbb{R}^3}\nabla\left(\frac{A_2(t)}{\rho^*}\right)\psi(t)\nabla\psi(t)\mathrm{d}x\right|+\left|\lambda\int_{\mathbb{R}^3}\nabla\left(\frac{A_2(t)}{\rho^*}\right)\psi(t)\nabla\cdot\psi(t)\mathrm{d}x\right|\right\}\equiv\\ K_1+K_2+K_3+K_4+K_5.\end{gathered} \tag{36}$$

Now, we estimate the right hand side of (36), using Sobolev inequality and Gagliard-Nirenberg inequality. By Hardy inequality, we have

$$K_1\leqslant\|(1+|x|)v^*\|_{L^\infty}\left\|\frac{\varphi(t)}{|x|}\right\|\|\nabla\varphi(t)\|\leqslant C\varepsilon\|\nabla\varphi(t)\|^2. \tag{37}$$

To estimate K_2, from (33) we have:

$$\begin{aligned}|R_1(t)|\leqslant C\Big\{&|\nabla v^*||\psi(t)|+(|v^*|+|\psi(t)|)|\nabla\psi(t)|+(|\nabla\rho^*|+|\nabla B^*|+|\nabla^2v^*|)|\varphi(t)|+\\ &|\nabla B^*||\omega(t)|+(|B^*|+|\omega|)|\nabla\omega(t)|+|\varphi||\nabla^2\psi(t)|\Big\}.\end{aligned}$$

Using integration by parts, we get

$$\left|\int_{\mathbb{R}^3} A_2(t)\psi(t)\varphi(t)\nabla^2\psi(t)\mathrm{d}x\right| \leqslant$$
$$\left|\int_{\mathbb{R}^3} \nabla\{A_2(t)\varphi(t)\psi(t)\}\nabla\psi(t)\mathrm{d}x\right| \leqslant$$
$$C\Big\{\|\varphi(t)\|_{L^\infty}\|\nabla\psi(t)\|^2 + \|(\nabla\rho^*, \nabla\varphi(t))\|_{L^3}\|w(t)\|_{L^6}\|\nabla\psi(t)\|\Big\} \leqslant$$
$$C\varepsilon\|\nabla\psi(t)\|^2,$$

so we have

$$K_2 \leqslant C\Big\{\|(1+|x|)^2\nabla v^*\|_{L^\infty}\left\|\frac{\psi(t)}{|x|}\right\|^2 + \|(1+|x|)v^*\|_{L^\infty}\|\nabla\psi(t)\|\left\|\frac{\psi(t)}{|x|}\right\| +$$
$$\|\psi(t)\|_{L^3}\|\nabla\psi(t)\|\|\psi(t)\|_{L^6} + \|(1+|x|)(\nabla\rho^*, \nabla B^*, \nabla^2 v^*)\|_{L^3}\left\|\frac{\varphi(t)}{|x|}\right\|\|\psi(t)\|_{L^6} +$$
$$\|(1+|x|)^2\nabla B^*\|_{L^\infty}\left\|\frac{\omega}{|x|}\right\|\left\|\frac{\psi}{|x|}\right\| + \|(1+|x|)B^*\|_{L^\infty}\|\nabla\omega(t)\|\left\|\frac{\psi(t)}{|x|}\right\| +$$
$$\|\omega(t)\|_{L^3}\|\nabla\omega(t)\|\|\psi(t)\|_{L^6} + \varepsilon\|\nabla\psi(t)\|^2\Big\} \leqslant$$
$$C\varepsilon\|(\nabla\varphi, \nabla\psi, \nabla\omega)(t)\|^2. \tag{38}$$

To estimate K_3, from (33) we have

$$|R_2(t)| \leqslant C\Big\{|\nabla v^*||\omega(t)| + |\nabla B^*||\psi(t)| + (|\omega| + |B^*|)|\nabla\psi(t)| + (|v^*| + |\psi|)|\nabla\omega(t)|\Big\},$$

so, we get

$$K_3 \leqslant C\Big\{\|(1+|x|)^2\nabla v^*\|_{L^\infty}\left\|\frac{\omega(t)}{|x|}\right\|^2 + \|(1+|x|)^2\nabla B^*\|_{L^\infty}\left\|\frac{\psi(t)}{|x|}\right\|\left\|\frac{\omega(t)}{|x|}\right\| +$$
$$\|(1+|x|)B^*\|_{L^\infty}\|\nabla\psi(t)\|\left\|\frac{\omega(t)}{|x|}\right\| + \|\omega(t)\|_{L^3}\|\nabla\psi(t)\|\|\omega(t)\|_{L^6} +$$
$$\|(1+|x|)v^*\|_{L^\infty}\|\nabla\omega(t)\|\left\|\frac{\omega(t)}{|x|}\right\| + \|\psi(t)\|_{L^3}\|\nabla\omega(t)\|\|\omega(t)\|_{L^6}\Big\} \leqslant$$
$$C\varepsilon\|(\nabla\varphi, \nabla\psi, \nabla\omega)(t)\|^2. \tag{39}$$

In order to estimate K_4, we use $(33)_1$:

$$2K_4 = \left|\int_{\mathbb{R}^3} A_{2t}(t)\psi(t)^2\mathrm{d}x\right| =$$
$$\left|\int_{\mathbb{R}^3} \tilde{A}_2(t)\varphi_t(t)\psi(t)^2\mathrm{d}x\right| =$$
$$\left|\int_{\mathbb{R}^3} \tilde{A}_2(t)\nabla\cdot\Big\{(\rho^* + \varphi(t))\psi(t) + v^*\varphi(t)\Big\}\psi(t)^2\mathrm{d}x\right| \leqslant$$
$$C\left|\int_{\mathbb{R}^3}\Big(\psi(t) + v^*\varphi(t)\Big)\Big(\nabla\{\psi(t)^2\} + \{\nabla\tilde{A}_2(t)\}\psi(t)^2\Big)\mathrm{d}x\right| \leqslant$$
$$C\Big\{\big(\|\psi(t)\|_{L^3} + \|v^*\|_{L^6}\|\varphi(t)\|_{L^6}\big)\|\nabla\psi(t)\|\|\psi(t)\|_{L^6} +$$
$$\|\big(\psi, \varphi\big)(t)\|_{L^6}\|\big(\nabla\rho^*, \nabla\varphi(t)\big)\|\|\psi(t)\|_{L^6}^2\Big\} \leqslant$$
$$C\varepsilon\|\nabla\psi(t)\|^2, \tag{40}$$

where $\tilde{A}_2(t)$ is defined by

$$\tilde{A}_2(t)=\frac{\rho^*+\varphi(t)}{P'\big(\rho^*+\varphi(t)\big)}\Big[2-\frac{P''\big(\rho^*+\varphi(t)\big)}{P'\big(\rho^*+\varphi(t)\big)}\big(\rho^*+\varphi(t)\big)\Big].$$

For K_5, we have the following estimate:

$$K_5\leqslant C\|\big(\nabla\rho^*,\nabla\varphi(t)\big)\|_{L^3}\|\nabla\psi(t)\|\|\psi(t)\|_{L^6}\leqslant C\varepsilon\|\nabla\psi(t)\|^2. \tag{41}$$

Combining (36)-(41), we obtain (35), if we choose $\varepsilon,\zeta>0$ small enough.

By the same discussion as in Lemma 6, we get the following high derivative estimate:

Lemma 7 Let $(\varphi,\psi,\omega)(t)$ be a solution to (33) as in Lemma 5 for any $t\in[0,t_1]$. Then there exist $\varepsilon_0>0,\zeta_0>0$ and $\alpha_k>0$ such that if $\varepsilon\leqslant\varepsilon_0$ and $\|(\varphi,\psi,\omega)(t)\|_3,\|\rho^*-\bar{\rho}\|_{I^4}+\|(v^*,B^*)\|_{J^5}\leqslant\varepsilon$, the following high order derivative estimate holds:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla^k\varphi(t)\|^2+\int_{\mathbb{R}^3}A_2(t)\big(\nabla^k\psi(t)\big)^2\mathrm{d}x+\int_{\mathbb{R}^3}\big(\nabla^k\omega(t)\big)^2\mathrm{d}x\Big]+\alpha_k\|(\nabla^{k+1}\psi(t),\nabla^{k+1}\omega(t))\|^2\leqslant\\ C(\varepsilon+\zeta)\|(\nabla\varphi(t),\nabla\psi(t),\nabla\omega(t))\|_{k-1}^2+C\zeta^{-1}\|(\nabla^k\psi(t),\nabla^k\omega(t))\|^2,\end{aligned} \tag{42}$$

where $1\leqslant k\leqslant 3$, $C>0$ is a constant depending on μ,λ and ν and any ζ with $0\leqslant\zeta\leqslant\zeta_0$.

Proof Using the Friedrichs mollifier, we may assume that $(\varphi,\psi,\omega)(t)\in C^0(0,t_1;H^\infty)\bigcap C^1(0,t_1;H^\infty)$. For any multi-index α with $1\leqslant|\alpha|\leqslant 3$, applying ∂_x^α to $(33)_1,(33)_2$ and $(33)_3$, multiplying the resultant equation by $\partial_x^\alpha\varphi(t),A_2\partial_x^\alpha\psi(t)$ and $\partial_x^\alpha\omega(t)$ respectively, and integrating the resulting formula with respect to x over $\mathbb{R}^3$, we have

$$\begin{gathered}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha\varphi(t)\|^2-\int_{\mathbb{R}^3}(\rho^*+\varphi(t))\partial_x^\alpha\psi(t)\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x=\int_{\mathbb{R}^3}\big(\partial_x^\alpha(v^*\varphi(t))+I_\alpha(t)\big)\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x,\\ \int_{\mathbb{R}^3}A_2(t)\partial_x^\alpha\psi_t(t)\partial_x^\alpha\psi(t)\mathrm{d}x+\int_{\mathbb{R}^3}(\rho^*+\varphi(t))\partial_x^\alpha\psi(t)\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x=\\ \int_{\mathbb{R}^3}A_2(t)\partial_x^\alpha\psi(t)\big(\partial_x^\alpha R_1(t)+J_\alpha(t)\big)\mathrm{d}x+\int_{\mathbb{R}^3}\frac{A_2(t)}{\rho^*}\partial_x^\alpha\Big\{\mu\Delta\psi(t)+\lambda\nabla(\nabla\cdot\varphi(t))\Big\}\partial_x^\alpha\psi(t)\mathrm{d}x,\\ \int_{\mathbb{R}^3}\partial_x^\alpha\omega_t(t)\partial_x^\alpha\omega(t)\mathrm{d}x=\int_{\mathbb{R}^3}\partial_x^\alpha R_2(t)\partial_x^\alpha\omega(t)\mathrm{d}x+\int_{\mathbb{R}^3}\nu\partial_x^\alpha\omega(t)\Delta\partial_x^\alpha\omega(t)\mathrm{d}x,\end{gathered}$$

where $I_\alpha(t),J_\alpha(t)$ are defined by

$$\begin{aligned}I_\alpha(t)=&\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial_x^{\alpha-\beta}(\rho^*+\varphi(t))\partial_x^\beta\psi(t),\\ J_\alpha(t)=&\sum_{\beta<\alpha}\binom{\alpha}{\beta}\Big[\Big(\partial_x^{\alpha-\beta}\frac{1}{\rho^*}\Big)\partial_x^\beta\Big\{\mu\Delta\psi(t)+\lambda\nabla\big(\nabla\cdot\psi(t)\big)\Big\}-\big(\partial_x^{\alpha-\beta}A_1(t)\big)\nabla\partial_x^\beta\varphi(t)\Big],\end{aligned}$$

Then, canceling the term $\int_{\mathbb{R}^3}(\rho^*+\varphi(t))\partial_x^\alpha\psi(t)\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x$ and by adding the above three formulas, writing the first term of the second formula and the third formula as follows:

$$\begin{aligned}&\int_{\mathbb{R}^3}A_2(t)\partial_x^\alpha\psi_t(t)\partial_x^\alpha\psi(t)\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}A_2(t)\big(\partial_x^\alpha\psi(t)\big)^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}A_{2t}(t)\big(\partial_x^\alpha\psi(t)\big)^2\mathrm{d}x,\\ &\int_{\mathbb{R}^3}\partial_x^\alpha\omega_t(t)\partial_x^\alpha\omega(t)\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\big(\partial_x^\alpha\omega(t)\big)^2\mathrm{d}x,\end{aligned}$$

and using integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\partial_x^\alpha\varphi(t)\|^2+\int_{\mathbb{R}^3}A_2(t)\big(\partial_x^\alpha\psi(t)\big)^2\mathrm{d}x+\int_{\mathbb{R}^3}\big(\partial_x^\alpha\omega(t)\big)^2dt\Big\}+\\
&\frac{\mu A_2(t)}{\rho^*}\|\nabla\partial_x^\alpha\psi(t)\|^2+\nu\|\nabla\partial_x^\alpha\omega(t)\|^2\leqslant\\
&\Big|\int_{\mathbb{R}^3}\partial_x^\alpha(v^*\varphi(t))\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x\Big|+\Big|\int_{\mathbb{R}^3}A_2(t)\partial_x^\alpha R_1(t)\partial_x^\alpha\psi(t)\mathrm{d}x\Big|+\Big|\int_{\mathbb{R}^3}\partial_x^\alpha R_2(t)\partial_x^\alpha\omega(t)\mathrm{d}x\Big|+\\
&\Big\{\Big|\int_{\mathbb{R}^3}I_\alpha(t)\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x\Big|+\Big|\int_{\mathbb{R}^3}A_2(t)J_\alpha(t)\partial_x^\alpha\psi(t)\mathrm{d}x\Big|\Big\}+\frac{1}{2}\Big|\int_{\mathbb{R}^3}A_{2t}\big(\partial_x^\alpha\psi(t)\big)^2\mathrm{d}x\Big|+\\
&\Big\{\Big|\mu\int_{\mathbb{R}^3}\nabla\Big(\frac{A_2(t)}{\rho^*}\Big)\partial_x^\alpha\psi(t)\nabla\partial_x^\alpha\psi(t)\mathrm{d}x\Big|+\Big|\lambda\int_{\mathbb{R}^3}\nabla\Big(\frac{A_2(t)}{\rho^*}\Big)\partial_x^\alpha\psi(t)\nabla\cdot\partial_x^\alpha\psi(t)\mathrm{d}x\Big|\Big\}\equiv\\
&K_1+K_2+K_3+K_4+K_5+K_6,
\end{aligned}\tag{43}$$

Now, we estimate the right hand side of (43), using Sobolev inequality and Gagliard-Nirenberg inequality. By integrating by parts, we show that

$$K_1\leqslant C\varepsilon\|\nabla\varphi(t)\|_{|\alpha|-1}^2.\tag{44}$$

To estimate K_2, we write $\partial_x^\alpha R_1$ in the form:

$$\partial_x^\alpha R_1(t)=-\frac{\varphi}{\rho^*(\rho^*+\varphi)}\Big[\mu\Delta\partial_x^\alpha\psi+\lambda\nabla(\nabla\cdot\partial_x^\alpha\psi)\Big]+F_\alpha(t).$$

Then we have the estimate for $F_\alpha(t)$:

$$\begin{aligned}
F_\alpha(t)\leqslant C\Big\{&|\nabla^{|\alpha|+1}v^*||\psi(t)|+\sum_{m=1}^{|\alpha|+1}|\nabla^m\psi(t)|+\sum_{m=1}^{|\alpha|+1}(|\nabla^m\rho^*|+|\nabla^m B^*|+|\nabla^{m+1}v^*|)|\varphi(t)|+\\
&\sum_{m=1}^{|\alpha|}|\nabla^m\varphi(t)|+\sum_{m=1}^{|\alpha|+1}|\nabla^m B^*||\omega(t)|+\sum_{m=1}^{|\alpha|+1}|\nabla^m\omega(t)|+D_F^{|\alpha|}(t)\Big\},
\end{aligned}\tag{45}$$

where

$$D_F^k(t)=\begin{cases}0 & \text{if } k=1\\ |\nabla^2\varphi(t)||\nabla^2\psi(t)| & \text{if } k=2\\ |\nabla^2\psi(t)||\nabla^3\varphi(t)|+(|\nabla^2\psi(t)|+|\nabla^3\psi(t)|)|\nabla^2\varphi(t)|+(|\nabla^4v^*|+ & \\ \quad|\nabla^3\rho^*|)|\nabla\varphi(t)|+(|\nabla^3\rho^*|+|\nabla^2\psi|)|\nabla^2\psi(t)|+|\nabla^2\omega(t)|^2 & \text{if } k=3\end{cases}.\tag{46}$$

Using Gagliard-Nirenberg inequality, we have

$$\begin{aligned}
&\|D_F^2(t)\|_{L^{3/2}}\leqslant\|\nabla^2\psi(t)\|_{L^6}\|\nabla^2\varphi(t)\|\leqslant C\varepsilon\|\nabla^2\varphi(t)\|,\\
&\|D_F^3(t)\|_{L^{3/2}}\leqslant\|\nabla^2\psi(t)\|_{L^6}\|\nabla^3\varphi(t)\|+\|\nabla^2\psi(t)\|_{L^6}\|\nabla^2\varphi(t)\|+\|\nabla^3\psi(t)\|\|\nabla^2\varphi(t)\|_{L^6}+\\
&(\|\nabla^4v^*\|,\|\nabla^3\rho^*\|)\|\nabla\varphi(t)\|_{L^6}+\|(\nabla^3\rho^*,\nabla^2\psi)\|\|\nabla^2\psi(t)\|_{L^6}+\\
&\|\nabla^2\omega(t)\|\|\nabla^2\omega(t)\|_{L^6}\leqslant\\
&C\varepsilon\{\|\nabla^2\varphi(t)\|_1+\|(\nabla^3\psi(t),\nabla^3\omega(t))\|\}.
\end{aligned}$$

Now we divide K_2 into the following two parts:

$$\begin{aligned}
K_2\leqslant&\Big|\int_{\mathbb{R}^3}A_2(t)F_\alpha(t)\partial_x^\alpha\psi(t)\mathrm{d}x\Big|+\Big|\int_{\mathbb{R}^3}\frac{\varphi(t)A_2(t)}{\rho^*(\rho^*+\varphi(t))}\Big[\mu\nabla\partial_x^\alpha\psi(t)+\lambda\nabla(\nabla\cdot\partial_x^\alpha\psi(t))\Big]\partial_x^\alpha\psi(t)\mathrm{d}x\Big|\equiv\\
&K_{21}+K_{22}.
\end{aligned}\tag{47}$$

Using integration by parts, we get an estimate for K_{22}:

$$K_{22} \leqslant \mu\left|\int_{\mathbb{R}^3} \nabla\left\{\frac{A_2(t)\varphi(t)}{\rho^*(\rho^*+\varphi(t))}\partial_x^\alpha\psi(t)\right\}\nabla\partial_x^\alpha\psi(t)\mathrm{d}x\right| + \lambda\left|\int_{\mathbb{R}^3} \nabla\cdot\left\{\frac{A_2(t)\varphi(t)}{\rho^*(\rho^*+\varphi(t))}\partial_x^\alpha\psi(t)\right\}\nabla\cdot\partial_x^\alpha\psi(t)\mathrm{d}x\right| \leqslant C\left\{\|\varphi(t)\|_{L^\infty}\|\nabla\partial_x^\alpha\psi(t)\|^2 + \|(\nabla\rho^*,\nabla\varphi(t))\|_{L^3}\|\partial_x^\alpha\psi(t)\|_{L^6}\|\nabla\partial_x^\alpha\psi(t)\|\right\} \leqslant C\varepsilon\|\nabla\partial_x^\alpha\psi(t)\|^2. \tag{48}$$

To estimate K_{21}, we use (45):

$$K_{21} \leqslant C\Bigg\{\|\nabla^{|\alpha|+1}v^*\|\|\psi(t)\|_{L^6}\|\nabla^{|\alpha|}\psi(t)\|_{L^3} + \sum_{m=1}^{|\alpha|+1}\|\nabla^m\psi(t)\|\|\nabla^\alpha\psi(t)\| + \sum_{m=1}^{|\alpha|+1}\|(\nabla^m\rho^*,\nabla^m B^*,\nabla^{m+1}v^*)\|\|\varphi(t)\|_{L^6}\|\nabla^\alpha\psi(t)\|_{L^3} + \sum_{m=1}^{|\alpha|}\|\nabla^m\varphi(t)\|\|\nabla^{|\alpha|}\psi(t)\| + \sum_{m=1}^{|\alpha|+1}\|\nabla^m B^*\|\|\omega(t)\|_{L^6}\|\nabla^\alpha\psi(t)\|_{L^3} + \sum_{m=1}^{|\alpha|+1}\|\nabla^m\omega(t)\|\|\nabla^{|\alpha|}\psi(t)\| + \|D_F^{|\alpha|}(t)\|_{L^{3/2}}\|\nabla^{|\alpha|}\psi(t)\|_{L^3}\Bigg\} \leqslant C(\varepsilon+\zeta)\big\{\|\nabla\varphi(t)\|_{|\alpha|-1}^2 + \|\big(\nabla\psi(t),\nabla\omega(t)\big)\|_{|\alpha|}^2\big\} + C\zeta^{-1}\|\nabla^{|\alpha|}\psi(t)\|^2. \tag{49}$$

Similar to K_2, we can estimate K_3 as follows:

$$K_3 \leqslant C(\varepsilon+\zeta)\big\{\|\nabla\varphi(t)\|_{|\alpha|-1}^2 + \|(\nabla\psi(t),\nabla\omega(t))\|_{|\alpha|}^2\big\} + C\zeta^{-1}\|\nabla^{|\alpha|}\omega(t)\|^2. \tag{50}$$

For $1 \leqslant |\alpha| \leqslant 3$, we can check that

$$K_4 \leqslant C\varepsilon\big\{\|\nabla\varphi(t)\|_{|\alpha|-1}^2 + \|\nabla\psi(t)\|_{|\alpha|}^2\big\}. \tag{51}$$

We note that the term $\big(I_\alpha(t),\nabla\partial_x^\alpha\varphi(t)\big)$ is estimated by integration by parts, and the case of $|\alpha| = 3$ can also be estimated using the following inequality:

$$\left\|\frac{\psi(t)}{1+|x|}\right\|_{L^\infty} \leqslant C\|\nabla\psi(t)\|_1$$

which follows from Sobolev inequality and Hardy inequality. In order to estimate K_5, we use $(3.1)_1$. Then

$$2K_5 = \left|\int_{\mathbb{R}^3} A_{2t}(t)\big(\partial_x^\alpha\psi(t)\big)^2\mathrm{d}x\right| = \left|\int_{\mathbb{R}^3}\tilde{A}_2(t)\varphi_t(t)\big(\partial_x^\alpha\psi(t)\big)^2\mathrm{d}x\right| = \left|\int_{\mathbb{R}^3}\tilde{A}_2(t)\partial_x^\alpha\psi(t)\cdot\partial_x^\alpha\psi(t)\nabla\cdot\{(\rho^*+\varphi(t))\psi(t)+v^*\varphi(t)\}\mathrm{d}x\right| \leqslant C\left|\int_{\mathbb{R}^3}\Big(\psi(t)+v^*\varphi(t)\Big)\Big(\nabla\{\partial_x^\alpha\psi(t)\cdot\partial_x^\alpha\psi(t)\} + \{\nabla\tilde{A}_2(t)\}\partial_x^\alpha\psi(t)\cdot\partial_x^\alpha\psi(t)\Big)\mathrm{d}x\right| \leqslant C\Big\{\big(\|\psi(t)\|_{L^3} + \|v^*\|_{L^6}\|\varphi(t)\|_{L^6}\big)\|\nabla\partial_x^\alpha\psi(t)\|\|\partial_x^\alpha\psi(t)\|_{L^6} + \|\big(\psi,\varphi\big)(t)\|_{L^6}\|\big(\nabla\rho^*,\nabla\varphi(t)\big)\|\|\partial_x^\alpha\psi(t)\|_{L^6}^2\Big\} \leqslant C\varepsilon\|\nabla\partial_x^\alpha\psi(t)\|^2, \tag{52}$$

where $\tilde{A}_2(t)$ is defined in Lemma 6.

For K_6, we have the following estimate:

$$K_6 \leqslant C\|\big(\nabla\rho^*, \nabla\varphi(t)\big)\|_{L^3}\|\nabla\partial_x^\alpha\psi(t)\|\|\partial_x^\alpha\psi(t)\|_{L^6} \leqslant C\varepsilon\|\nabla\partial_x^\alpha\psi(t)\|^2. \tag{53}$$

Combining (43)-(53), we obtain (42), if we choose $\varepsilon, \zeta > 0$ small enough.

3.2 The estimate for ψ_t, ω_t and their derivatives up to $\nabla^2\psi_t(t)$, $\nabla^2\omega_t(t)$

Lemma 8 Let $(\varphi, \psi, \omega)(t)$ be a solution to (33) as Lemma 5 for any $t \in [0, t_1]$. Then there exist $\varepsilon_0 > 0$ and $\gamma_1 > 0$ such that if $\varepsilon \leqslant \varepsilon_0$ and $\|(\varphi, \psi, \omega)(t)\|_3, \|\rho^* - \bar{\rho}\|_{I^4} + \|(v^*, B^*)\|_{J^5} \leqslant \varepsilon$, the following estimate holds when $k = 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\psi(t)\nabla\varphi(t)\mathrm{d}x + \gamma_1\|(\psi_t(t), \omega_t(t))\|^2 \leqslant C\big\{\varepsilon\|\nabla\varphi(t)\|^2 + \|(\nabla\psi(t), \nabla\omega(t))\|_1^2\big\}, \tag{54}$$

where $C > 0$ is a constant depending only on μ, λ and ν.

Proof Using the Friedrichs mollifier, we may assume that $(\varphi, \psi, \omega)(t) \in C^0(0, t_1; H^\infty)\bigcap C^1(0, t_1; H^\infty)$. Multiply $(3.1)_2$ by $A_1(t)^{-1}$, we have

$$\frac{\psi_t(t)}{A_1(t)} + \nabla\varphi(t) = \frac{1}{\rho^* A_1(t)}\big[\mu\Delta\psi(t) + \lambda\nabla\big(\nabla\cdot\psi(t)\big)\big] + \frac{f(t)}{A_1(t)}.$$

Then, multiplying the resultant equation by $\psi_t(t)$, we have

$$\int_{\mathbb{R}^3}\psi_t(t)\nabla\varphi(t)\mathrm{d}x + \int_{\mathbb{R}^3}\frac{1}{A_1(t)}\big(\psi_t(t)\big)^2\mathrm{d}x = \int_{\mathbb{R}^3}\left\{\frac{1}{\rho^* A_1(t)}\big[\mu\Delta\psi(t) + \lambda\nabla\big(\nabla\cdot\psi(t)\big)\big] + \frac{R_1(t)}{A_1(t)}\right\}\psi_t(t)\mathrm{d}x,$$

and multiplying $(33)_3$ by $\omega_t(t)$, we have

$$\|\omega_t(t)\|^2 = \int_{\mathbb{R}^3}\nu\omega_t(t)\Delta\omega(t)\mathrm{d}x + \int_{\mathbb{R}^3}R_2(t)\omega_t(t)\mathrm{d}x,$$

where

$$\int_{\mathbb{R}^3}\psi_t(t)\nabla\varphi(t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\psi(t)\nabla\varphi(t)\mathrm{d}x + \int_{\mathbb{R}^3}\varphi_t(t)\nabla\cdot\psi(t)\mathrm{d}x.$$

Therefore, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\psi(t)\nabla\varphi(t)\mathrm{d}x + \frac{1}{A_1(t)}\|\psi_t(t)\|^2 + \|\omega_t(t)\|^2 \leqslant \\ &\left|\int_{\mathbb{R}^3}\left\{\frac{1}{\rho^* A_1(t)}\big[\mu\Delta\psi(t) + \lambda\nabla\big(\nabla\cdot\psi(t)\big)\big]\right\}\psi_t(t)\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3}\nu\omega_t(t)\Delta\omega(t)\mathrm{d}x\right| + \\ &\left|\int_{\mathbb{R}^3}\left\{\frac{R_1(t)}{A_1(t)}\right\}\psi_t(t)\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3}R_2(t)\omega_t(t)\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3}\varphi_t(t)\nabla\cdot\psi(t)\mathrm{d}x\right| \equiv \\ &K_1 + K_2 + K_3 + K_4 + K_5.\end{aligned} \tag{55}$$

Now we estimate the right hand side of (55), using Sobolev inequality and Gagliard-Nirenberg inequality. We can easily check that

$$K_1 \leqslant \zeta\|\nabla\psi_t(t)\|^2 + C\zeta^{-1}\|\nabla^2\psi(t)\|^2, \tag{56}$$

$$K_2 \leqslant \zeta\|\nabla\omega_t(t)\|^2 + C\zeta^{-1}\|\nabla^2\omega(t)\|^2. \tag{57}$$

From (33) as (38) and (39), we have

$$\begin{aligned} K_3 \leqslant C\big\{&\|\varphi(t)\|_{L^\infty}\|\nabla^2\psi(t)\| + \|\nabla v^*\|_{L^3}\|\psi(t)\|_{L^6} + \\ &\|\big(v^*,\psi(t)\big)\|_{L^\infty}\|\nabla\psi(t)\| + \|\big(\nabla\rho^*,\nabla B^*,\nabla^2 v^*\big)\|_{L^3}\|\varphi(t)\|_{L^6} + \\ &\|\nabla B^*\|_{L^3}\|\omega(t)\|_{L^6} + \|(B^*,\omega)\|_{L^\infty}\|\nabla\omega\|\big\}\|\psi_t(t)\| \leqslant \\ &C\varepsilon\big\{\|\nabla\varphi(t)\|^2 + \|\nabla\psi(t)\|_1^2 + \|\nabla\omega(t)\|^2 + \|\psi_t(t)\|^2\big\}, \end{aligned} \tag{58}$$

and

$$\begin{aligned} K_4 \leqslant &\|\nabla v^*\|_{L^3}\|\omega(t)\|_{L^6}\|\omega_t(t)\| + \|\nabla B^*\|_{L^3}\|\psi(t)\|_{L^6}\|\omega_t(t)\| + \\ &\|\big(\omega(t),B^*\big)\|_{L^\infty}\|\nabla\psi(t)\|\|\omega_t(t)\| + \|\big(\psi(t),v^*\big)\|_{L^\infty}\|\nabla\omega(t)\|\|\omega_t(t)\| \leqslant \\ &C\varepsilon\big\{\|\big(\nabla\psi(t),\nabla\omega(t)\big)\|^2 + \|\omega_t(t)\|^2\big\}. \end{aligned} \tag{59}$$

In order to estimate K_5, we substitute $(33)_1$ into φ_t as in (40), we have

$$\begin{aligned} K_5 \leqslant &\left|\int_{\mathbb{R}^3}\nabla\cdot\big\{\big(\rho^*+\varphi(t)\big)\psi(t)\big\}\nabla\cdot\psi(t)\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3}v^*\varphi(t)\nabla\big(\nabla\cdot\psi(t)\big)\mathrm{d}x\right| \leqslant \\ &C\bigg\{\|\big(\nabla\rho^*,\nabla\varphi(t)\big)\|_{L^3}\|\psi(t)\|_{L^6}\|\nabla\psi(t)\| + \|\nabla\psi(t)\|^2 + \\ &\|(1+|x|)v^*\|_{L^\infty}\left\|\frac{\varphi(t)}{|x|}\right\|\|\nabla^2\psi(t)\|\bigg\} \leqslant \\ &C\varepsilon\|(\nabla\varphi(t),\nabla^2\psi(t))\|^2 + C\|\nabla\psi(t)\|^2, \end{aligned} \tag{60}$$

Combining (55)-(60), we get (54) for $\varepsilon,\zeta > 0$ small enough. This completes the proof of Lemma 8.

With similar discussion as in Lemma 8, we get the following estimate for $2 \leqslant k \leqslant 3$:

Lemma 9 Let $(\varphi,\psi,\omega)(t)$ be a solution to (33) as in Lemma 5 for any $t \in [0,t_1]$. Then there exist $\varepsilon_0 > 0$ and $\gamma_k > 0$ such that if $\varepsilon \leqslant \varepsilon_0$ and $\|(\varphi,\psi,\omega)(t)\|_3, \|\rho^*-\bar\rho\|_{I^4} + \|(v^*,B^*)\|_{J^5} \leqslant \varepsilon$, the following estimate holds when $2 \leqslant k \leqslant 3$:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^{k-1}\psi(t)\nabla^k\varphi(t)\mathrm{d}x + \gamma_k\|(\nabla^{k-1}\psi_t(t),\nabla^{k-1}\omega_t(t))\|^2 \leqslant \\ &C\big\{\|\nabla\varphi(t)\|_{k-2}^2 + \|(\nabla\psi(t),\nabla\omega(t))\|_k^2\big\} + C\varepsilon\|\nabla^{k-2}\psi_t(t)\|^2, \end{aligned} \tag{61}$$

where $C > 0$ is a constant depending only on μ,λ and ν.

3.3 The estimate for $\nabla\varphi(t)$ and its derivatives up to $\nabla^3\varphi(t)$

Lemma 10 Let $(\varphi,\psi,\omega)(t)$ be a solution to (33) as in Lemma 5 for any $t \in [0,t_1]$. Then there exist $\varepsilon_0 > 0$ and $\zeta_0 > 0$ such that if $\varepsilon \leqslant \varepsilon_0$ and $\|(\varphi,\psi,\omega)(t)\|_3, \|\rho^*-\bar\rho\|_{I^4} + \|(v^*,B^*)\|_{J^5} \leqslant \varepsilon$, the following estimates hold:

$$\|\nabla\varphi(t)\|^2 \leqslant C\big\{\|\nabla\psi(t)\|_1^2 + \|(\nabla\omega(t),\psi_t(t))\|^2\big\}, \tag{62}$$

$$\|\nabla^k\varphi(t)\|^2 \leqslant C\big\{\|\nabla\varphi(t)\|_{k-2}^2 + \|\nabla\psi(t)\|_k^2 + \|\nabla\omega(t)\|_{k-1}^2 + \|\nabla^{k-1}\psi_t(t)\|^2\big\}. \tag{63}$$

for $2 \leqslant k \leqslant 3$, where $C > 0$ is a constant depending only on μ,λ and ν.

Proof Using the Friedrichs mollifier, we may assume that $(\varphi,\psi,\omega)(t)\in C^0(0,t_1;H^\infty)\bigcap C^1(0,t_1;H^\infty)$. For any multi-index α with $0\leqslant|\alpha|\leqslant 2$, applying ∂_x^α to $(33)_2$ and multiplying the resultant equation by $\nabla\partial_x^\alpha\varphi(t)$, we have

$$\begin{gathered}\int_{\mathbb{R}^3}A_1(t)\big(\nabla\partial_x^\alpha\varphi(t)\big)^2\mathrm{d}x\leqslant\\ \left|\int_{\mathbb{R}^3}\partial_x^\alpha\psi_t(t)\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}\partial_x^\alpha\Big\{\frac{1}{\rho^*}\big[\mu\Delta\psi(t)+\lambda\nabla\big(\nabla\cdot\psi(t)\big)\big]\Big\}\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x\right|+\\ \left|\int_{\mathbb{R}^3}I_\alpha(t)\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}\partial_x^\alpha R_1(t)\nabla\partial_x^\alpha\varphi(t)\mathrm{d}x\right|\equiv\\ K_1+K_2+K_3+K_4,\end{gathered}\tag{64}$$

where $I_\alpha(t)$ is defined by

$$I_\alpha(t)=\sum_{\beta<\alpha}\binom{\alpha}{\beta}\big(\partial_x^{\alpha-\beta}A_1(t)\big)\nabla\partial_x^\beta\varphi(t).$$

It follows immediately from Sobolev inequality that

$$\begin{aligned}&K_1\leqslant\zeta\|\nabla^{|\alpha|+1}\varphi(t)\|^2+C\zeta^{-1}\|\nabla^{|\alpha|}\psi_t(t)\|^2,\\ &K_2\leqslant\zeta\|\nabla^{|\alpha|+1}\varphi(t)\|^2+C\zeta^{-1}\|\nabla^2\psi(t)\|_{|\alpha|}^2,\\ &K_3\leqslant C\varepsilon\|\nabla\varphi(t)\|_{|\alpha|}^2.\end{aligned}\tag{65}$$

To estimate K_4, if $\alpha=0$ as (38), we have

$$K_4\leqslant\zeta\|\nabla\varphi(t)\|^2+C\zeta^{-1}\big\{\|\nabla\varphi(t)\|^2+\|\nabla\psi(t)\|_1^2+\|\nabla\omega(t)\|^2\big\},\tag{66}$$

if $1\leqslant\alpha\leqslant 2$, as (45)-(49), we have

$$K_4\leqslant\zeta\|\nabla^{|\alpha|+1}\varphi(t)\|^2+C\zeta^{-1}\big\{\|\nabla\varphi(t)\|_{|\alpha|-1}^2+\|\nabla\psi(t)\|_{|\alpha|+1}^2+\|\nabla\omega(t)\|_{|\alpha|}^2\big\}.\tag{67}$$

Combining (64)-(67), we obtain (62)-(63) if we take $\varepsilon,\zeta>0$ small enough. This completes the proof of Lemma 10.

3.4 Summarization of the lemmas in the previous three subsections

Let $(\varphi,\psi,\omega)(t)$ be a solution to (33) as in Lemma 5 locally in time with $t\in[0,t_1]$. Furthermore, we suppose that $\|(\varphi,\psi,\omega)(t)\|_3,\|\rho^*-\bar{\rho}\|_{I^4}+\|(v^*,B^*)\|_{J^5}\leqslant\varepsilon$, where $\varepsilon>0$ is small enough such that at least we can use the results obtained in Lemmas 6-10. We use the notation:

$$[\varphi,\psi,\omega]=\|\varphi(t)\|^2+\int_{\mathbb{R}^3}A_2(t)\psi(t)^2\mathrm{d}x+\int_{\mathbb{R}^3}\omega(t)^2\mathrm{d}x,$$

where $A_2(t)$ is defined as in Lemma 6.

Summing up (35), (42) with $k=1$, (54) and (62) (after multiplying (42), (54) and (62) with small numbers respectively), we have

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{m=0}^{1}\alpha_m\big[\nabla^m\varphi,\nabla^m\psi,\nabla^m\omega\big]+\beta_1\int_{\mathbb{R}^3}\psi\nabla\varphi\mathrm{d}x\Big\}+\\ \|\nabla\varphi\|^2+\|(\nabla\psi,\nabla\omega)\|_1^2+\|(\psi_t,\omega_t)\|^2\leqslant 0,\end{gathered}\tag{68}$$

if we take $\varepsilon, \zeta > 0$ sufficiently small. Here and hereafter $\alpha_m, \beta_m > 0$ are constants depending only on μ, λ and ν. Then summing up (42), (61), (63) with $k = 2$ and (68), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{m=0}^{2}\alpha_m\left[\nabla^m\varphi, \nabla^m\psi, \nabla^m\omega\right] + \sum_{m=1}^{2}\beta_m\int_{\mathbb{R}^3}\nabla^{m-1}\psi\nabla^m\varphi\mathrm{d}x\right\} + \|\nabla\varphi\|_1^2 + \|(\nabla\psi, \nabla\omega)\|_2^2 + \|(\psi_t, \omega_t)\|_1^2 \leqslant 0. \tag{69}$$

Also, by (42), (61), (63) with $k = 3$ and (69), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{m=0}^{3}\alpha_m\left[\nabla^m\varphi, \nabla^m\psi, \nabla^m\omega\right] + \sum_{m=1}^{3}\beta_m\int_{\mathbb{R}^3}\nabla^{m-1}\psi\nabla^m\varphi\mathrm{d}x\right\} + \|\nabla\varphi\|_2^2 + \|(\nabla\psi, \nabla\omega)\|_3^2 + \|(\psi_t, \omega_t)\|_2^2 \leqslant 0, \tag{70}$$

for any $t \in [0, t_1]$. Then integrating (70) with respect to t over $[0, t_1]$ implies that

$$N[\varphi, \psi, \omega](t) + \int_0^t \|\nabla\varphi(s)\|_2^2 + \|(\nabla\psi, \nabla\omega)(s)\|_3^2 + \|(\psi_t, \omega_t)(s)\|_2^2\mathrm{d}s \leqslant N[\varphi, \psi, \omega](0), \tag{71}$$

where $N[\varphi, \psi, \omega](s)$ is defined by

$$N[\varphi, \psi, \omega](s) = \sum_{m=0}^{3}\alpha_m\left[\nabla^m\varphi, \nabla^m\psi, \nabla^m\omega\right](s) + \sum_{m=1}^{3}\beta_m\int_{\mathbb{R}^3}\nabla^{m-1}\psi(s)\nabla^m\varphi(s)\mathrm{d}x,$$

for any $s > 0$.

Let us denote $A_0 = \min_{\bar\rho/2\leqslant s\leqslant 3\bar\rho/2}\left\{A_2(s), 1\right\}$ and $\tilde{A}_0 = \max_{\bar\rho/2\leqslant s\leqslant 3\bar\rho/2}\left\{A_2(s), 1\right\}$. Since we may assume without loss of generality that $\alpha_k \leqslant \alpha_{k-1}$ and $\beta_k \leqslant \alpha_k\min\{A_0, 1\}/4$ for $k = 1, 2, 3$, it follows from simple calculation that

$$\frac{\alpha_3}{4}A_0\|(\varphi, \psi, \omega)(s)\|_3^2 \leqslant N[\varphi, \psi, \omega](s) \leqslant 2\alpha_0\tilde{A}_0\|(\varphi, \psi, \omega)(s)\|_3^2 \tag{72}$$

for each $s \in [0, t_1]$. Combining (71) and (72), we obtain (34), which completes the proof of Lemma 5.

Hence, by Lemmas 4 and 5, we finally arrive at the conclusion of Theorem 2.

References:

[1] Kulikovskiy A G, Lyubimov G A. Magnetohydrodynamics, Reading[M]. New York: MA Addison-Wesley, 1965.

[2] Laudau L D, Lifshitz E M. Electrodynamics of Continuous Media[M]. 2nd ed. New York: Pergamon, 1984.

[3] Polovin R V, Demutskii V P. Fundamentals of magnetohydrodynamics[M]. New York: Consultants Bureau, 1990.

[4] Sermange M, Temam R. Some mathematical questions related to the MHD equations[J]. Comm Pure Appl Math, 1983, 36:635-664.

[5] Chen G Q, Wang D H. Global solution of nonlinear magnetohydrodynamics with large inital data[J]. J Differ Equ, 2002, 182:344-376.

[6] Fan J, Yu W. Global variational solutions to the compressible magnetohydrodynamic equations[J]. Nonlin Analysis, 2008, 69:3637-3660.

[7] Fan J, Yu W. Strong solutions to the compressible magnetohydrodynamic equations with vacuum[J]. Nonlin Analysis, 2009, 70:392-409.

[8] Hao C C. Well-posedness to the compressible viscous magnetohydrodynamic system[J]. Nonlinear Anal: Real World Appl, 2011, 12:2962-2972.

[9] Hoff D, Tsyganov E. Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics[J]. Z Angew Math Phys, 2005, 56:791-804.

[10] Hu X P, Wang D H. Global solutions to the three-dimensional full compressible magnetohydrodynamic flows[J]. Comm Math Phys, 2008, 283:255-284.

[11] Hu X P, Wang D H. Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows[J]. Arch Ration Mech Anal, 2010, 197:203-238.

[12] Kawashima S. Smooth global solutions for two-dimensional equations of electromagnetofluid dynamics[J]. Proc Jpn Acad, 1982, A62:384-387.

[13] Kawashima S, Okada M. Smooth global solutions for the one-dimensional equations in magnetohydrodynamics[J]. Jap J Appl Math, 1984, 1:207-222.

[14] Li F C, Yu H J. Optimal decay rate of classical solution to the compressible magnetohyrtodynamic equations[J]. Proc Roy Soc Edinburgh, 2011, 141A:109-126.

[15] Umeda T, Kawashima S, Shizutu Y. On the decay of solutions to the linearized equations of electromagnetofluid dynamics[J]. Jpn J Appl Math, 1984, 1:435-457.

[16] Wang D H. Large solutions to the initial-boundary value problem for planar magnetohydrodynamics[J]. SIAM J Appl Math, 2003, 63:1424-1441.

[17] Danchin R. Low Mach number limit for compressible flows with periodic boundary conditions[J]. Amer J Math, 2002, 124:1153-1219.

[18] Fan J, Jiang S, Nakamura G. Vanishing shear viscosity limit in the magnetohydrodynamic equations[J]. Comm Math Phys, 2007, 270:691-708.

[19] Freistühlera H, Trakhinin Y. On the vicous and inviscid stability of magnetohydrodynamic shock wave[J]. Phys D 2008, 237:3030-3037.

[20] Hu X P, Wang D H. Low Mach number limit to the three-dimensional full compressible magnetohydrodynamic flows[J]. SIAM J Math Anal, 2009, 41:1272-1294.

[21] Jiang S, Ju Q C, Li F C. Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions[J]. Comm Math Phys, 2010, 297:371-400.

[22] Jiang S, Ju Q C, Li F C. Incompressible limit of the compressible magnetohydrodynamic equations with vanishing viscosity coefficients[J]. SIAM J Math Anal, 2010, 43:2539-2553.

[23] Shibata Y, Tanaka K. On the steady flow of compressible viscous fluid and its stability with respect to initial disturbance[J]. J Math Soc Japan, 2003, 55:797-826.

[24] Novotny A, Padula M. Physically reasonable solutions to steady compressible Navier-Skokes equations in 3D-exterior domains($v_\infty = 0$)[J]. J Math Kyoto Univ, 1996, 36:389-422.

[25] Matsumura A, Nishida T. The initial-value problem for the equation of motion of viscous and heat-conductive gases[J]. J Math Kyoto Univ, 1980, 20:67-104.

[26] Adams R. Sobolev spaces[M]. New York: Academic Press, 1985.

[27] Duan R J, Liu H, Ukai S, Yang T. Optimal $L^p - L^q$ convergence rates for the compressible Navier-Stokes equations with potential force[J]. J Differ Equ, 2007, 238:220-233.

[28] Duan R J, Ukai S, Yang T, Zhao H J. Optimal convergence rates for the compressible Navier-Stokes equations with potential force[J]. Math Models Methods Appl Sci, 2007, 17:1-22.

[29] Kobayashi T. Some estimates of solutions for the equations of motion of compressible viscous fluid in the three-dimensional exterior domain[J]. J Differ Equ, 2002, 184:587-619.

[30] Matsumura A, Nishida T. The initial-value problem for the equation of motion of compressible viscous and heat-conductive fluids[J]. Proc Jpn Acad, 1979, A55:337-342.

[31] Ponce G. Global existence of small solution to a class of nolinear evolution equations[J]. Nonlinear Anal, 1985, 9:399-418.

[32] Taylor M E. Partial differential equation (III): Nolinear equation[M]. Berlin: Springer, 1996.

[33] Ukai S, Yang T, Zhao H J. Convergence rates for the compressible Navier-Stokes equations with potential force[J]. J Hyperbolic Diff Eqns, 2006, 3:561-574.

[34] Deckelinick K. Decay estimates for the compressible Navier-Stokes equations in unbounded domains[J]. Math Z 1992, 209:115-130.

[35] Hoff D, Zumbrun K. Multidimensional diffusion waves for the Navier-Stokes equations of compressible flow[J]. Indiana Univ Math J, 1995, 44:604-676.

[36] Kagei Y, Kobayashi T. Asymptotic behavior of solutions of the compressible Navier-Stokes equations on the half space[J]. Arch Ration Mech Anal, 2005, 177:231-330.

[37] Kobayashi T, Shibata Y. Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbb{R}^3$[J]. Comm Math Phys, 1999, 200:621-659.

[38] Lions P L. Mathematical Topics in Fluid Mechanics, vol. 2[M]. Oxford: Compressible Models Press, 1998.

[39] Liu T P, Wang W K. The pointwise estimates of diffusion waves for the Navier-Stokes equations in odd multi-dimensions[J]. Comm Math Phys, 1998, 196:145-173.

[40] Matsumura A, Nishida T. Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids[J]. Comm Math Phys, 1983, 89(4):445–464.

[41] Matsumura A, Nishida T. Exterior stationary problems for the equation of motion of compresible viscous and heat-conductive fluids[C]//Dafermos M, Ladas G, Papanicolau G. Lect Notes Pure Appl Math, vol 118. New York: Dekker, 1989:473-479.

[42] Novotny A, Padula M. Physically reasonable solutions to steady compressible Naviver-Skokes equations in 3D-exterior domains($v_\infty \neq 0$)[J]. Math Ann, 1997, 308:439-489.

[43] Novotny A V, Straskraba I. Introduction to the Mathematical Theory of Compressible Flow[M]. Oxford: Oxford Univ Press, 2004.

[44] Shibata Y, Tanaka K. Rate of convergence of non-stationary flow to the steady flow of compressible viscous fluid[J]. Comput Math Appl, 2007, 53:605-623.

[45] Tanaka K. A stability of steady flow compressible viscous fluid with respect to initial disturbance($v_\infty \neq 0$)[J]. Math Methods Appl Sci 2006, 29:1451-1466.

[46] Tanaka K. Rate of convergence of non-stationary flow to the steady flow of compressible viscous fluid[J]. preprint.

[47] Qian J Z, Yin H. On the stationary solution of the full compressible Navier-Stokes equations and its stability with respect to initial disturbance[J]. J Differ Equ, 2007, 237:225-256.

可压缩磁流体动力学方程 (MHD) 稳态解的存在性及相对于初值扰动的稳定性

吴仁超

(上海师范大学 数理学院，上海 200234)

摘　要：研究了有外力影响的三维可压缩粘性磁流体动力学方程 (MHD) 解的存在性．首先推导出稳态解的存在性，其次研究当稳态解相对于初值扰动时，方程整体解的存在性.

关键词：磁流体方程；稳态解；光滑解；能量估计

(责任编辑：包震宇)