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Stability analysis of nonlinear delay differential-algebraic equations and of the implicit euler methods

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Abstract: We consider the stability and asymptotic stability of a class of nonlinear delay differential-algebraic equations and of the implicit Euler methods. Some sufficient conditions for the stability and asymptotic stability of the equations are given. These conditions can be applied conveniently to nonlinear equations. We also show that the implicit Euler methods are stable and asymptotically stable.

Key words: nonlinear differential-algebraic equation; delay; implicit Euler method

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1 Introduction

In recent years, much research has been focused on numerical solutions of systems of differential-algebraic equations (DAEs). These systems can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical systems, power-systems, chemical process simulation, and optimal control. In some cases, time delays appear in variables of unknown functions so that the Differential-Algebraic Equations (DAEs) are converted to Delay Differential-Algebraic Equations (DDAEs). Delay-DAEs (DDAEs), which have both delays and algebraic constraints, arise frequently in circuit simulation and power system, Among numerous results on DDAE systems, there are few achievements on nonlinear systems. The solution of a nonlinear system depends on a nonlinear manifold of a product space as well as on consistent initial valued-vectors over a space of continuous functions. It is pointed in [1-2] that research on nonlinear DDAEs is more complicated and still remains investigated.

In this paper, we investigate a class of nonlinear DDAE systems, and show the conditions under which the analytical solutions are stable and asymptotically stable. Similarly, the implicit Euler methods retain the asymptotic behaviors.

2 Asymptotic behavior of a class of nonlinear DDAEs

2.1 Stability of analytical solutions of nonlinear DDAEs

In this subsection, we consider the following nonlinear system of delay differential-algebraic equations,

$$u'(t) = f(u(t), u(t-\tau), v(t), v(t-\tau)), t > 0, (\tau > 0), \tag{1}$$

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$$0 = \varphi(u(t), u(t-\tau), v(t)), t > 0. \quad (2)$$

According to [3] the assumption that φ_v is nonsingular allows one to solve the constraint equations (2) for $v(t)$ (using the implicit function theorem), yielding

$$v(t) = g(u(t), u(t-\tau)). \quad (3)$$

By substituting (3) into (1) we obtain the DODE

$$u'(t) = f(u(t), u(t-\tau), g(u(t), u(t-\tau)), g(u(t-\tau), u(t-2\tau))). \quad (4)$$

Thus, the DDAE (1), (2) are stable if the DODE (4) is stable. Note that if all the delay terms are present in this retarded DODE, then the initial conditions need to be defined for t on $[-2\tau, 0]$, In this paper we investigate the following nonlinear DDAEs

$$u'(t) = f(t, u(t), u(t-\tau), v(t), v(t-\tau)), t > 0, (\tau > 0), \quad (5)$$

$$0 = \varphi(u(t), u(t-\tau), v(t)), t > 0, \quad (6)$$

$$u(t) = \varphi_1(t), v(t) = \phi_1(t), -\tau \leqslant t \leqslant 0, \quad (7)$$

$$u(t) = \varphi_2(t), v(t) = \phi_2(t), -2\tau \leqslant t \leqslant -\tau. \quad (8)$$

and its perturbed equations

$$\tilde{u}'(t) = f(t, \tilde{u}(t), \tilde{u}(t-\tau), \tilde{v}(t), \tilde{v}(t-\tau)), t > 0, (\tau > 0), \quad (9)$$

$$0 = \varphi(\tilde{u}(t), \tilde{u}(t-\tau), \tilde{v}(t)), t > 0, \quad (10)$$

$$\tilde{u}(t) = \tilde{\varphi}_1(t), \tilde{v}(t) = \tilde{\phi}_1(t), -\tau \leqslant t \leqslant 0, \quad (11)$$

$$\tilde{u}(t) = \tilde{\varphi}_2(t), \tilde{v}(t) = \tilde{\phi}_2(t), -2\tau \leqslant t \leqslant -\tau. \quad (12)$$

Definition 2.1[4] The system (5)-(8) are said to be stable. if the following inequalities are satisfied:

$$\|u(t) - \tilde{u}(t)\| \leqslant \max_{-2\tau \leqslant t \leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|, \forall t \geqslant 0, \quad (13)$$

$$\|v(t) - \tilde{v}(t)\| \leqslant M \max_{-2\tau \leqslant t \leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|, \forall t \geqslant 0. \quad (14)$$

where $M > 0$ is a constant,

$$\Phi(t) = \begin{cases} \varphi_1(t), & -\tau \leqslant t \leqslant 0, \\ \varphi_2(t), & -2\tau \leqslant t \leqslant -\tau, \end{cases} \quad \text{and} \quad \tilde{\Phi}(t) = \begin{cases} \tilde{\varphi}_1(t), & -\tau \leqslant t \leqslant 0, \\ \tilde{\varphi}_2(t), & -2\tau \leqslant t \leqslant -\tau. \end{cases} \quad (15)$$

To study the stability of the DDAE (5)-(8), it is necessary to introduce several lemmas.

Lemma 2.1[4-5] Consider the following initial value problem

$$\chi'(t) = a(t)\chi(t) + \eta(t), \chi(0) = \chi_0, t > 0, \quad (16)$$

where $a(t)$, $\eta(t)$ are continuous functions of t when $t \geqslant 0$, $\mathrm{Re}(a(t)) < 0$. Then the solution of the initial value problem (16) satisfies

$$|\chi(t)| \leqslant \max\left\{ |\chi(0)|, \max_{0 \leqslant s \leqslant t} \frac{|\eta(s)|}{-\mathrm{Re}(a(s))} \right\}, \quad (17)$$

for all $t \geqslant 0$.

Now we require that f, φ satisfy the following Lipschitz conditions (1)-(4):

(1) $\langle f(t, u, u_\tau, v, v_\tau) - f(t, \tilde{u}, u_\tau, v, v_\tau), u - \tilde{u} \rangle \leqslant \sigma(t) \|u - \tilde{u}\|^2$,

(2) $\|f(t, u, u_\tau, v, v_\tau) - f(t, u, \tilde{u}_\tau, v, v_\tau)\| \leqslant \gamma_1(t) \|u_\tau - \tilde{u}_\tau\|$, $\|f(t, u, u_\tau, v, v_\tau) - f(t, u, u_\tau, \tilde{v}, v_\tau)\| \leqslant \gamma_2(t) \|v - \tilde{v}\|$, $\|f(t, u, u_\tau, v, v_\tau) - f(t, u, u_\tau, v, \tilde{v}_\tau)\| \leqslant \gamma_3(t) \|v_\tau - \tilde{v}_\tau\|$,

(3) φ_v is nonsingular, so that for $g(u, v)$ in (3) there exists $L > 0$ and $K > 0$ such that

$$\|g(u, v) - g(\tilde{u}, v)\| \leqslant L\|u - \tilde{u}\|, \|g(u, v) - g(u, \tilde{v})\| \leqslant K\|v - \tilde{v}\|.$$

(4) $\sigma(t)<0,\gamma_1(t)+(L+K)\gamma_2(t)+(L+K)\gamma_3(t)\leqslant-\sigma(t),\forall t\geqslant 0$.

Theorem 2.1 If f and φ in (5) - (6) satisfy the conditions (1) - (4), then (5) - (6) is stable.

Proof Let $u=u(t), u_\tau=u(t-\tau), v=v(t), v_\tau=v(t-\tau)$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-\tilde{u}\|^2=\langle f(t,u,u_\tau,v,v_\tau)-f(t,\tilde{u},\tilde{u}_\tau,\tilde{v},\tilde{v}_\tau),u-\tilde{u}\rangle=$$

$$\langle f(t,u,u_\tau,v,v_\tau)-f(t,\tilde{u},u_\tau,v,v_\tau),u-\tilde{u}\rangle+\langle f(t,\tilde{u},u_\tau,v,v_\tau)-f(t,\tilde{u},\tilde{u}_\tau,v,v_\tau),u-\tilde{u}\rangle+$$

$$\langle f(t,\tilde{u},\tilde{u}_\tau,v,v_\tau)-f(t,\tilde{u},\tilde{u}_\tau,\tilde{v},v_\tau),u-\tilde{u}\rangle+\langle f(t,\tilde{u},\tilde{u}_\tau,\tilde{v},v_\tau)-f(t,\tilde{u},\tilde{u}_\tau,\tilde{v},\tilde{v}_\tau),u-\tilde{u}\rangle. \quad (18)$$

Let $Y(t)=\|u(t)-\tilde{u}(t)\|$. Applying the Schwartz inequality on the right side of (18) and noting the conditions (1), (2), we get

$$Y(t)Y'(t)\leqslant\sigma(t)Y^2(t)+\gamma_1(t)\|u_\tau-\tilde{u}_\tau\|Y(t)+\gamma_2(t)\|v-\tilde{v}\|Y(t)+\gamma_3(t)\|v_\tau-\tilde{v}_\tau\|Y(t). \quad (19)$$

From the condition (3), we get

$$\|v-\tilde{v}\|=\|g(u,u_\tau)-g(\tilde{u},\tilde{u}_\tau)\|=\|g(u,u_\tau)-g(\tilde{u},u_\tau)+g(\tilde{u},u_\tau)-g(\tilde{u},\tilde{u}_\tau)\|\leqslant$$

$$\|g(u,u_\tau)-g(\tilde{u},u_\tau\|+\|g(\tilde{u},u_\tau)-g(\tilde{u},\tilde{u}_\tau)\|\leqslant L\|u-\tilde{u}\|+K\|u_\tau-\tilde{u}_\tau\|.$$

Thus

$$\|v(t)-\tilde{v}(t)\|=\|g(u,u_\tau)-g(\tilde{u},\tilde{u}_\tau)\|\leqslant L\|u(t)-\tilde{u}(t)\|+K\|u_\tau-\tilde{u}_\tau\|,t\geqslant 0. \quad (20)$$

According to (20), (19) becomes

$$Y'(t)\leqslant(\sigma(t)+L\gamma_2(t))Y(t)+\eta(t).$$

$$\eta(t)=(\gamma_1(t)+K\gamma_2(t))\|u_\tau-\tilde{u}_\tau\|+\gamma_3(t)\|v_\tau-\tilde{v}_\tau\|.$$

Consider the following initial value problem of differential equation:

$$\tilde{Y}'(t)=(\sigma(t)+L\gamma_2(t))\tilde{Y}(t)+\eta(t),t>0, \quad (21)$$

$$\tilde{Y}(0)=\|\Phi(0)-\Phi(0)\|. \quad (22)$$

Using the Lemma 2.1, the solution of (21) - (22) satisfies

$$\tilde{Y}(t)\leqslant\max\left\{\|\Phi(0)-\tilde{\Phi}(0)\|,\max_{0\leqslant s\leqslant t}\frac{\eta(s)}{-(\sigma(s)+L\gamma_2(s))}\right\},t\geqslant 0.$$

Let $Y(t)=\gamma_1(t)+K\gamma_2(t)+L\gamma_3(t)$. It becomes

$$\tilde{Y}(t)\leqslant\max\|\Phi(0)-\tilde{\Phi}(0)\|,$$

$$\max_{0\leqslant s\leqslant t}\frac{Y(s)\|u(s-\tau)-\tilde{u}(s-\tau)\|+K\gamma_3(s)\|u(s-2\tau)-\tilde{u}(s-2\tau)\|}{-(\sigma(s)+L\gamma_2(s))}\Big\}, \quad (23)$$

If $t\in[0,\tau]\Rightarrow s-\tau\in[-\tau,0], s-2\tau\in[-2\tau,-\tau]$. Note the consistent initial value functions $\Phi(t)$, $\tilde{\Phi}(t)$. (23) becomes

$$\tilde{Y}(t)\leqslant\max\{\|\Phi(0)-\tilde{\Phi}(0)\|,$$

$$\max_{0\leqslant s\leqslant t}\frac{Y(s)\|\Phi(s-\tau)-\tilde{\Phi}(s-\tau)\|+K\gamma_3(s)\|\Phi(s-2\tau)-\tilde{\Phi}(s-2\tau)\|}{-(\sigma(s)+L\gamma_2(s))}\Big\}. \quad (24)$$

According to the condition (4), we get

$$\tilde{Y}(t)\leqslant\max_{-2\tau\leqslant t\leqslant}\|\Phi(t)-\tilde{\Phi}(t)\|,t\in[0,\tau]. \quad (25)$$

It is easy to verify that

$$Y(t)\leqslant\tilde{Y}(t),\forall t\geqslant 0.$$

Thus

$$Y(t) \leqslant \max_{-2\tau\leqslant t\leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|, t \in [0,\tau]. \tag{26}$$

If $t\in[\tau,2\tau] \Rightarrow s-\tau\in[0,\tau], s-2\tau\in[-\tau,0]$. Similarly, we also get

$$Y(t) \leqslant \max_{-2\tau\leqslant t\leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|, t \in [\tau,2\tau]. \tag{27}$$

Applying mathematical induction, we conclude that

$$Y(t) \leqslant \max_{-2\tau\leqslant t\leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|, \forall t \geqslant 0. \tag{28}$$

Therefore

$$\|u(t) - \tilde{u}(t)\| \leqslant \max_{-2\tau\leqslant t\leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|, \forall t \geqslant 0.$$

By (20)

$$\|v(t) - \tilde{v}(t)\| \leqslant M \max_{-2\tau\leqslant t\leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|, \forall t \geqslant 0.$$

2.2 Asymptotic stability of analytical solutions of nonlinear DDAEs

Definition 2.2 The delay differential-algebraic equations (5) – (8) are asymptotically stable if and only if for all consistent initial value functions $\phi, \tilde{\phi}(t), \varphi, \tilde{\varphi}(t)$, the solutions $\{u(t), v(t)\}$, $\{\tilde{u}(t), \tilde{v}(t)\}$ satisfy

$$\lim_{t\to\infty}\|u(t) - \tilde{u}(t)\| = 0, \lim_{t\to\infty}\|v(t) - \tilde{v}(t)\| = 0.$$

In order to study the asymptotic stability of (5) – (8), the following Lemma is needed.

Lemma 2.2[4-5] Suppose that a non-negative function $Z(t)$ satisfies

$$Z'(t) = \omega(t)Z(t) + \gamma_1(t)Z(t-\tau) + \gamma_2(t)Z(t-2\tau), t>0, \tau>0, Z(t) = \varphi(t), t\leqslant 0,$$

where $\varphi(t)\geqslant 0, \omega(t), \gamma_1(t), \gamma_2(t)$, are given functions, and

$$\gamma_1(t) + \gamma_2(t) \leqslant -q\omega(t), 0\leqslant q<1, \forall t\geqslant 0, \omega(t)\leqslant -\beta<0, \forall t\geqslant 0.$$

Then $Z(t)\to 0\ (t\to\infty)$.

Apply (20) to (19), we have

$$\begin{aligned} Y'(t) &\leqslant \sigma(t)Y(t) + L\gamma_2(t)\|u(t) - \tilde{u}(t)\| + \eta(t) \leqslant \\ &(\sigma(t) + L\gamma_2(t))Y(t) + Y(t)Y(t-\tau) + K\gamma_3(t)Y(t-2\tau). \end{aligned} \tag{29}$$

Theorem 2.2 If (5) – (8) satisfies conditions (1), (2), (3) and (4)′

$$\sigma(t) + L\gamma_2(t) \leqslant -\beta < 0, \sup_{t\geqslant 0} \frac{\gamma_1(t) + K\gamma_2(t) + (K+L)\gamma_3(t)}{-(\sigma(t) + L\gamma_2(t))} = q, 0\leqslant q<1,$$

then (5) – (8) is asymptotically stable.

Proof Consider the initial value problem of the delay differential equations

$$\tilde{Z}'(t) = \omega(t)\tilde{Z}(t) + Y(t)\tilde{Z}(t-\tau) + Y_1(t)\tilde{Z}(t-2\tau), t>0,$$

$$\tilde{Z}(t) = \|\Phi(t) - \tilde{\Phi}(t)\|, -2\tau\leqslant t\leqslant 0,$$

where

$$\omega(t) = \sigma(t) + L\gamma_2(t),\ Y_1(t) = K\gamma_3(t). \tag{30}$$

From (4)′, all the conditions of Lemma (2.2) are satisfied, so

$$\lim_{t\to\infty}\tilde{Z}(t) = 0.$$

Note (29). It is easy to verify

$$Y(t) \leqslant \tilde{Z}(t), \forall t\geqslant 0,$$

Therefore,

$$\lim_{t\to\infty} Y(t) = \lim_{t\to\infty}\|u(t) - \tilde{u}(t)\| = 0,$$

and the theorem is proved.

3 The stability and asymptotic stability Applying Implicit Euler Methods

Definition 3.1 A numerical method for solving DDAEs is said to be stable, if for all consistent initial value functions $\phi,\tilde{\phi},\phi,\tilde{\phi}$ and each step $h>0$, the solution sequences $\{u_n,v_n\}$, $\{\tilde{u}_n,\tilde{v}_n\}$ for (5)-(8) and (9)-(12) in which f,g satisfy conditions (1) ~ (4), satisfy

$$\|u_n - \tilde{u}_n\| \leqslant \max_{-2\tau\leqslant t\leqslant}\|\Phi(t) - \tilde{\Phi}(t)\|, n = 0,1,2,\cdots,$$

$$\|v_n - \tilde{v}_n\| \leqslant L\max_{-2\tau\leqslant t\leqslant}\|\Phi(t) - \tilde{\Phi}(t)\|, n = 0,1,2,\cdots.$$

Consider the initial value problem of the ordinary differential equations

$$x'(t) = f(t,x(t)), t>0, \tag{31}$$

$$x(0) = x_0. \tag{32}$$

The implicit Euler methods can be written as:

$$x_{n+1} = x_n + hf(t_{n+1},x_{n+1}), n = 0,1,2,\cdots, \tag{33}$$

$$x_0 = x(0), \tag{34}$$

where $x_n \sim x(t_n)$, $h>0$ is the step size. Note (1)-(3), to solve (5)-(8) by (33)-(34), we get

$$u_{n+1} = u_n + hf(t_{n+1},u_{n+1},u_{n+1-m},v_{n+1},v_{n+1-m}), n = 0,1,2,\cdots, \tag{35}$$

$$v_{n+1} = g(u_{n+1},u_{n+1-m}), \tag{36}$$

$$u_n = \phi_1(t_n), v_n = \phi_1(t_n), -m\leqslant n\leqslant 0, \tag{37}$$

$$u_n = \phi_2(t_n), v_n = \phi_2(t_n), -2m\leqslant n\leqslant -m, (mh=\tau, m\geqslant 1). \tag{38}$$

The Perturbations of (35)-(38) are

$$\tilde{u}_{n+1} = \tilde{u}_n + hf(t_{n+1},\tilde{u}_{n+1},\tilde{u}_{n+1-m},\tilde{v}_{n+1},\tilde{v}_{n+1-m}), n = 0,1,2,\cdots, \tag{39}$$

$$\tilde{v}_{n+1} = g(\tilde{u}_{n+1},\tilde{u}_{n+1-m}), \tag{40}$$

$$\tilde{u}_n = \tilde{\phi}_1(t_n), \tilde{v}_n = \tilde{\phi}_1(t_n), -m\leqslant n\leqslant 0, \tag{41}$$

$$\tilde{u}_n = \tilde{\phi}_2(t_n), \tilde{v}_n = \tilde{\phi}_2(t_n), -2m\leqslant n\leqslant -m, (mh=\tau, m\geqslant 1). \tag{42}$$

Theorem 3.1 The implicit Euler methods are stable for DDAEs.

Proof Let $\bar{V}_n = u_n - \tilde{u}_n$. Substituting yields

$$\begin{aligned}\bar{V}_{n+1} = {} & \bar{V}_n + h\{f(t_{n+1},u_{n+1},u_{n+1-m},v_{n+1},v_{n+1-m}) - f(t_{n+1},\tilde{u}_{n+1},\tilde{u}_{n+1-m},\tilde{v}_{n+1},\tilde{v}_{n+1-m})\} = \\ & \bar{V}_n + hf(t_{n+1},u_{n+1},u_{n+1-m},v_{n+1},v_{n+1-m}) - hf(t_{n+1},\tilde{u}_{n+1},u_{n+1-m},v_{n+1},v_{n+1-m}) + \cdots + \\ & hf(t_{n+1},\tilde{u}_{n+1},\tilde{u}_{n+1-m},\tilde{v}_{n+1},v_{n+1-m}) - hf(t_{n+1},\tilde{u}_{n+1},\tilde{u}_{n+1-m},\tilde{v}_{n+1},\tilde{v}_{n+1-m}).\end{aligned} \tag{43}$$

Applying the Schwartz theorem and the condition (1)-(2), we obtain

$$\begin{aligned}\|\bar{V}_{n+1}\|^2 \leqslant {} & \|\bar{V}_n\|\cdot\|\bar{V}_{n+1}\| + h\sigma(t_{n+1})\|\bar{V}_{n+1}\|^2 + \\ & h\gamma_1(t_{n+1})\|\bar{V}_{n+1-m}\|\cdot\|\bar{V}_{n+1}\| + h\gamma_2(t_{n+1})\|v_{n+1} - \tilde{v}_{n+1}\|\cdot\|\bar{V}_{n+1}\| + \\ & h\gamma_3(t_{n+1})\|v_{n+1-m} - \tilde{v}_{n+1-m}\|\cdot\|\bar{V}_{n+1}\|.\end{aligned} \tag{44}$$

Assume that $\|\bar{V}_{n+1}\|\neq 0$ and note (36), (40) and (16).

$$v_{n+1} = g(u_{n+1},u_{n+1-m}),\ \tilde{v}_{n+1} = g(\tilde{u}_{n+1},\tilde{u}_{n+1-m}), \tag{45}$$

$$\|v_{n+1} - \tilde{v}_{n+1}\| \leqslant L\|u_{n+1} - \tilde{u}_{n+1}\| + K\|u_{n+1-m} - \tilde{u}_{n+1-m}\|, n = 0,1,2,\cdots, \tag{46}$$

(44) is divided by $\|\bar{V}_{n+1}\|$, Noting the consistency of the initial value function, we get

$$\|\bar{V}_{n+1}\| \leqslant \frac{\|\bar{V}_n\| + hY(t_{n+1})\|\bar{V}_{n+1-m}\| + hK\gamma_3(t_{n+1})\|\bar{V}_{n+1-2m}\|}{1 - h\sigma(t_{n+1}) - hL\gamma_2(t_{n+1})}, n = 0,1,\cdots. \tag{47}$$

In (45), let $0 \leqslant n \leqslant m-1$, and note the initial value function. We have

$$\|\bar{V}_{n+1}\| \leqslant \max_{-2\tau \leqslant t \leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|. \tag{48}$$

When $n \geqslant m$, applying mathematical induction, we find that (48) is true for all $n \geqslant 0$. As for $\|v_n - \tilde{v}_n\|$, just see (46),

$$\|v_n - \tilde{v}_n\| \leqslant M \max_{-2\tau \leqslant t \leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|.$$

In the rest of this section, we study the asymptotic stability of Euler methods. First, we have the following definition.

Definition 3.2[17] A numerical method for solving the DDAE (5) - (8) is said to be asymptotically stable if and only if when it is applied to (5) - (8) and (9) - (12), for any consistent initial value functions $\phi(t)$, $\tilde{\phi}(t)$, $\varphi(t)$, $\tilde{\varphi}(t)$ and any step size $h > 0$, its approximation solution sequences $\{u_n, v_n\}$, $\{\tilde{u}_n, \tilde{v}_n\}$ satisfy the following equalities,

$$\lim_{n\to\infty}\|u_n - \tilde{u}_n\| = 0, \lim_{n\to\infty}\|v_n - \tilde{v}_n\| = 0.$$

where f, g satisfy conditions (1), (2), (3)′, (4).

Theorem 3.2 The implicit Euler methods are asymptotically stable for DDAEs.

Proof Noting (47) and (26), we have

$$\|\bar{V}_{n+1}\| \leqslant \frac{\|\bar{V}_n\| + hY_1(t_{n+1})\|\bar{V}_{n+1-m}\| + hY_2(t_{n+1})\|\bar{V}_{n+1-2m}\|}{1 - h\omega(t_{n+1})}, n = 0,1,2\cdots.$$

Let $0 \leqslant n \leqslant m-1$ in the above inequality, we get

$$\|\bar{V}_{n+1}\| \leqslant \max_{t \geqslant 0}\left(\frac{1 + hY_1(t) + hY_2(t)}{1 - h\omega(t)}\right) \cdot \max_{-2\tau \leqslant t \leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|.$$

Note condition (4)′,

$$\frac{1 + hY_1(t) + hY_2(t)}{1 + h|\omega(t)|} \leqslant \frac{1 + hq|\omega(t)|}{1 + h|\omega(t)|} \leqslant \frac{1 + h\beta q}{1 + h\beta}: = p < 1, \forall t \geqslant 0.$$

Therefore, when $0 \leqslant n \leqslant m-1$,

$$\|\bar{V}_{n+1}\| \leqslant p \max_{-2\tau \leqslant t \leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|.$$

For the case $n = m$

$$\|\bar{V}_{m+1}\| \leqslant \frac{\|\bar{V}_m\| + hY(t_{m+1})\|\bar{V}_1\| + hY_1(t_{m+1})\|\bar{V}_{1-m}\|}{1 - h\omega(t_{m+1})} \leqslant p^2 \max_{-2\tau \leqslant t \leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|.$$

For the case $rm \leqslant n \leqslant (r+1)m - 1$, it can be shown by induction that

$$\|\bar{V}_{m+1}\| \leqslant p^{r+1} \max_{-2\tau \leqslant t \leqslant} \|\Phi(t) - \tilde{\Phi}(t)\|.$$

When $n \to \infty$, $r \to \infty$,

$$\|\bar{V}_{m+1}\| \to 0, (n \to \infty).$$

Thus,

$$\|u_{n+1} - \tilde{u}_{n+1}\| \to 0, (n \to \infty), \|v_{n+1} - \tilde{v}_{n+1}\| \to 0, (n \to \infty).$$

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非线性延时微分代数方程和隐式欧拉方法的稳定性分析

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摘　要: 考虑了一类非线性延时微分代数方程隐式欧拉方法的稳定性和渐近稳定性,给出了稳定和渐近稳定的一些充分条件. 这些条件便于应用到非线性方程. 也证明了隐式欧拉方法是稳定和渐近稳定的.

关键词: 非线性微分代数方程; 延迟; 隐式欧拉方法

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