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Improvement algorithms for discrete-time control systems based on the extension and localization principles

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Abstract: In this paper, we present iterative improvement algorithms for free-end and constrained discrete-time systems. These algorithms are based on the extension and localization principles. In the case of constrained end problem, a method based on approximation of reachable set is proposed. Algorithm procedures are illustrated with examples.

Key words: optimal control; discrete system; extension principle; localization principle; reachable set; iterative optimization

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基于广义和局部化法则的时间离散控制系统的改进算法

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摘　要：分别讨论了右端自由和右端固定的时间离散控制系统的改进算法．这些算法是基于广义和局部化法则，对右端固定的情景，提出了一种基于允许集的逼近方法．并用实例说明了算法的有效性.

关键词：最优控制；离散系统；广义法则；局部法则；容许集；迭代优化

1 Introduction

Discrete dynamical models form an important class of mathematical models that can describe a wide range of real world systems and corresponding to them optimal control problems. There exists a large amount of literature

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on methods of optimal control and improvements. In general, these existing methods are based on such fundamental results as the maximum principle of Pontryagin[1], the dynamic programming method of Bellman[2], and the sufficient conditions of optimality of Krotov[3]. It should be noted that the majority of optimal control methods, both exact and approximate, are designed for continuous time systems. For general discrete systems, especially nonlinear, a set of such methods is more limited due to the fact that in the general case there is no analogue of the Pontryagin maximum principle for the discrete-time systems[4−6].

Discrete models can be used to solve optimal control problems (including continuous ones with prior discretization of variables) through direct application of non-linear programming methods[7−10]. In terms of optimal control theory the results based on the Bellman principle of optimality and general sufficient conditions of optimality of Krotov are much more developed. They include, in particular, local optimality conditions and iterative improvement methods for discrete and hybrid systems[11−14].

This paper proposes iterative methods of local optimization for free-end and constrained discrete systems using the extension and localization principles[15−17] and description of the reachable set[15, 18−19]. The algorithm procedures are demonstrated by examples.

2 Problem formulation and preliminary results

Consider a control system described by the following equations:

$$x\left(t+1\right)=f\left(t,x\left(t\right),u\left(t\right)\right),\quad \mathbf{T}=\left\{t_I,t_I+1,\ldots,t_F-1\right\}, \tag{1}$$

$$x\left(t\right)\in\mathbf{X}\left(t\right)\subset\mathbb{X}\left(t\right),\quad u\left(t\right)\in\mathbf{U}\left(t,x\left(t\right)\right)\subset\mathbb{U}\left(t\right),$$

$$\left(t_I,x\left(t_I\right),t_F,x\left(t_F\right)\right)\in\mathbf{\Gamma}\subset\mathbb{G},$$

$$\mathbb{G}=\left\{\gamma:\ \tau,\vartheta\in\mathbb{N},\ \xi\left(\tau\right)\in\mathbb{X}\left(\tau\right),\ \chi\left(\vartheta\right)\in\mathbb{X}\left(\vartheta\right)\right\},$$

$$I=F\left(t_I,x\left(t_I\right),t_F,x\left(t_F\right)\right),\ \ F:\mathbb{G}\to\mathbb{R},$$

where $\gamma=\left(\tau,\xi\left(\tau\right),\vartheta,\chi\left(\vartheta\right)\right)$, $\mathbb{X}\left(t\right),\mathbb{U}\left(t\right)$ are the basic sets (spaces) of arbitrary nature, in general different for different t. Let $\mathbf{T}$, $x(t_I)$ be given, $x(t_F)\in\mathbf{X}_F$. Assume that functions $f(t,x,u)$ and $F(x)$ have good analytical properties.

We denote by $\mathbf{D}$ the set of its possible solutions $m=\left(\mathbf{T},x\left(t\right),u\left(t\right)\right)$. It is required to find a sequence $\left\{m_s\right\}\subset\mathbf{D}$ such that $I\left(m_s\right)\to I_*=\inf\limits_{\mathbf{D}}I$.

The essence of the extension principle is the replacement of the problem $\left(\mathbf{D},I\right)$ by an analogous problem $\left(\mathbf{E},L\right)$, in some sense simpler, which also gives a solution of the original problem. Extensions $\left(\mathbf{E},L\right)$ are defined as follows. The discrete chain (1) is excluded, thereby the set $\mathbf{E}$ is uniquely determined. For any $t\in\mathbb{N}$ an arbitrary functional

$$\varphi:\ \mathbb{X}\left(t\right)\to\mathbb{R}$$

is defined. The functional L is determined by

$$L=F\left(t_I,x\left(t_I\right),t_F,x\left(t_F\right)\right)+\varphi\left(t_F,x\left(t_F\right)\right)-$$

$$\varphi\left(t_I, x\left(t_I\right)\right) - \left(\sum_{t_I}^{t_F-1}\left(\varphi\left(t+1, f\left(t, x\left(t\right), u\left(t\right)\right)\right) - \varphi\left(t, x\left(t\right)\right)\right)\right).$$

Obviously, $L = I$ for $m \in \mathbf{D}$ when constraint (1) is satisfied. Indeed, regrouping the terms gives

$$\sum_{t_I}^{t_F-1}\left(\varphi\left(t+1, x\left(t+1\right)\right) - \varphi\left(t, x\left(t\right)\right)\right) = \varphi\left(t_F, x\left(t_F\right)\right) - \varphi\left(t_I, x\left(t_I\right)\right),$$

$$L = F\left(t_I, x\left(t_I\right), t_F, x\left(t_F\right)\right) = I.$$

Denote

$$R\left(t, x\left(t\right), u\left(t\right)\right) = \varphi\left(t+1, f\left(t, x\left(t\right)\right), u\left(t\right)\right) - \varphi\left(t, x\left(t\right)\right),$$

$$\mu\left(t\right) = \sup_{u \in \mathbf{U}(t,x(t)),\ x \in \mathbf{X}(t)} R\left(t, x(t), u(t)\right),$$

$$G\left(\gamma\right) = F\left(\gamma\right) + \varphi(\vartheta, \chi(\vartheta)) - \varphi\left(\tau, \xi(\tau)\right) - \sum_{\tau}^{\vartheta-1}\mu\left(t\right),$$

$$l = \inf G\left(\gamma\right), \quad \gamma \in \mathbf{\Gamma} \cap \left\{\gamma :\ \xi\left(\tau\right) \in \mathbf{X}\left(\tau\right),\ \chi\left(\vartheta\right) \in \mathbf{X}\left(\vartheta\right)\right\}.$$

Then

$$L = G\left(t_I, x\left(t_I\right), t_F, x\left(t_F\right)\right) + \sum_{t_I}^{t_F-1}\left(\mu\left(t\right) - R\left(t, x\left(t\right), u\left(t\right)\right)\right).$$

These expressions are used to formulate the following general proposition comprising Krotov sufficient optimality conditions for the problem being considered. Denote by $\mathbf{T}_{\max}$ any segment of $\mathbb{N}$ which covers the projection of $\mathbf{\Gamma}$ on t axis.

Theorem 1 (Gurman V.I.[15]) Let there be a sequence $\{m_s\} \subset \mathbf{D}$ and a sequence $\varphi_q\left(t\right)$, $t \in \mathbf{T}_{\max}$ such that

1) $R\left(t, x_s\left(t\right), u_s\left(t\right)\right) - \mu_q\left(t\right) \to 0, \quad t \in \mathbf{T}_{\max}$;

2) $G\left(\left(t_I, x\left(t_I\right), t_F, x\left(t_F\right)\right)_s\right) = l_q \to 0$.

Then

1) $I\left(m^{II}\right) < I\left(m^{I}\right)$, if $L_q\left(m^{II}\right) < L_q\left(m^{I}\right)$;

2) $I\left(m_s\right) - \inf\limits_{\mathbf{D}} I \leqslant \Delta_{qs} = L_q\left(m_s\right) - l_q$;

3) $\{m_s\}$ is minimizing: $I\left(m_s\right) \to I_* = \inf\limits_{\mathbf{D}} I$.

This proposition follows immediately from the extension principle. The theorem reduces optimization problems with discrete recurrent constraints to problems without such constraints, or, in more detail, to mathematical programming problems (minimizing $G\left(\gamma\right)$ and maximizing $R\left(t, x, u\right)$ with respect to x and u for various given values of t).

3 Control improvement for free-end discrete problem

3.1 Derivation of the improvement algorithm

Let an element $m^I = (x(t),\ u(t))^I \in \mathbf{D}$ be given. The problem is to improve m^I, that is, to find $m^{II} \in \mathbf{D}$ such that $I(m^{II}) < I(m^I)$. Repeating to solve this improvement problem sequentially, we get an iterative process aimed to minimize the functional I.

The localization principle[15, 17] is to reduce the improvement problem to a global optimization for a simplified model which is valid in the vicinity of improving element. According to the localization method, we introduce a family of functionals

$$I_\alpha = F_\alpha\left(x^0\left(t_F\right), x\left(t_F\right)\right) = (1-\alpha)\, F\left(x\left(t_F\right)\right) + \alpha x^0\left(t_F\right), \quad 0 \leqslant \alpha \leqslant 1,$$
$$x^0\left(t+1\right) = x^0\left(t\right) + \frac{1}{2}|du|^2, \quad x^0\left(t_I\right) = 0, \quad du = u - u^I(t). \tag{2}$$

The improvement of m^I is achieved via minimization of I_α for the system (1) and (2).

Following the extension principle, we introduce a scalar function

$$\varphi = \nu\left(t\right) + \psi^{\mathrm{T}}\left(t\right) dx + \frac{1}{2} dx^{\mathrm{T}} \sigma\left(t\right) dx, \quad dx = x - x^I\left(t\right),$$

where $\nu(t)$, $\psi(t)$ and $\sigma(t)$ are a scalar, an n-vector and an $(n \times n)$-matrix functions respectively, and the generalized Lagrangian

$$L = G\left(\tilde{x}\left(t_F\right)\right) - \sum_{t_I}^{t_F-1} R\left(t, \tilde{x}\left(t\right), u\left(t\right)\right),$$
$$R\left(t, \tilde{x}, u\right) = \varphi(t+1, \tilde{f}(t, \tilde{x}, u)) - \varphi(t, \tilde{x}),\, G\left(\tilde{x}\right) = F_\alpha(\tilde{x}_F) + \varphi(t_F, \tilde{x}_F) - \varphi(t_I, \tilde{x}_I),$$
$$\tilde{x} = \left(x, x^0\right), \quad \tilde{f}\left(t, \tilde{x}, u\right) = \left(f\left(t, x, u\right), x^0 + \frac{1}{2}|du|^2\right).$$

According to the extension principle, the problem of minimum of I_α for the system (1) and (2) can be reduced to the minimization problem of L without the discrete chains.

Equations w.r.t. unknown Taylor elements of φ can be derived with the help of Taylor approximations of constructions R and G in the neighborhood of $(x^I(t), u^I(t))$. The following basic relations for the improvement algorithm are obtained.

$$\nu(t) = \nu(t+1) - \frac{1}{2}(H_u^I)^{\mathrm{T}} N^{-1} H_u^I, \quad \nu(t_F) = -(1-\alpha) F\left(x^I\left(t_F\right)\right), \tag{3}$$

$$\psi(t) = H_x^I - \left(H_{xu}^I + (f_x^I)^{\mathrm{T}} \sigma(t+1) f_u^I\right) N^{-1} H_u^I, \quad \psi(t_F) = -(1-\alpha) F_x(x^I(t_F)), \tag{4}$$

$$\sigma(t) = H_{xx}^I + (f_x^I)^{\mathrm{T}} \sigma(t+1) f_x^I - (H_{xu}^I + (f_x^I)^{\mathrm{T}} \sigma(t+1) f_u^I) N^{-1} ((f_x^I)^{\mathrm{T}} \sigma(t+1) f_u^I + H_{xu}^I)^{\mathrm{T}}, \tag{5}$$
$$\sigma(t_F) = -(1-\alpha) F_{xx}\left(x^I(t_F)\right),$$
$$d\tilde{u} = -N^{-1}(H_u^I + (H_{xu}^I + f_x^I \sigma(t+1) f_u^I)^{\mathrm{T}} dx), \tag{6}$$

where

$$H = \psi^{\mathrm{T}}(t+1) f(t, x, u) - x^0(t) - \frac{1}{2}|du|^2,\ N = H_{uu}^I,\ (f_x^I)^{\mathrm{T}} \sigma(t+1) f_u^I < 0.$$

3.2 Iterative improvement algorithm

Thus the following iterative procedure is obtained.

1. Solve Eq. (1) "from the left to the right" for $x(t_I) = x_I$, $u = u^I(t)$, $t \in \mathbf{T}$, obtain $x^I(t)$.
2. Solve Eqs. (3)–(6) "from the right to the left". Check conditions $\alpha = 0$, $H_u^I = 0$, $N < 0$ for every t at the same time. If they hold then the improvement process finishes. If condition $H_u^I = 0$ fails then go to 3, otherwise go to 4.
3. Repeat 2 for different α, choose such α_* that the condition $N < 0$ holds in the entire set $\mathbf{T}$, and $\Delta_\alpha = I(m^I) - I(m_\alpha)$ is maximized. Take $m_{\alpha_*} = m^{II}$, $u^{II}(t)$ for $u^I(t)$ and go to 1.
4. If the condition $N < 0$ fails at a certain point $t^* \in \mathbf{T}$ then choose such α that this condition would hold in the entire set $\mathbf{T}$, except point t_I in which $\det N = 0$. Calculate ν, ψ, σ.
5. Solve Eq. (1) repeatedly for different $\epsilon > 0$ with the initial condition $x(t_I) = x_I$ and $u(t) = u^I(t) + du(t)$, where $du(t_I)$ is taken such that $N(t_I)du = 0$, $|du| < \epsilon$, and following Eq. (6) for $t > t_I$. Find ϵ_* maximizing $\Delta_\epsilon = I(m^I) - I(m_\epsilon)$, then take m_{ϵ_*} for m^{II}, $u^{II}(t)$ for $u^I(t)$ and go to 1 to start the next iteration.

The overall iterative process is stopped when $\alpha_* \approx 0$ and $m^I \approx m^{II}$ with the required accuracy. The final result is at least a local minimum and a local approximate synthesis of optimal control.

Remark 1 If the conditions that $\alpha = 0$, $H_u^I = 0$, $N < 0$ are satisfied, then $m^I = \left(x^I(t), u^I(t)\right)$ delivers a local maximum to $R(t, x, u)$ w.r.t. (x, u) for all $t \in \mathbf{T}$, a local minimum to $G(x)$ and a local minimum to the functional. If it proves to be not only local, but also global maximum of $R(t, x, u)$ w.r.t. u in the neighborhood of $x^I(t)$ then the control field $\bar{u}(t, x) = u^I(t) + d\tilde{u}(t, dx)$ is approximate local optimal control synthesis in an ϵ-neighborhood of the local minimal.

Remark 2 Assume $\sigma = 0$ in Eqs. (3), (4), and (6). Then the above algorithm of the second order becomes the first order one:

$$
\begin{gathered}
\nu(t) = \nu(t+1) - \frac{1}{2}(H_u^I)^{\mathrm{T}} N^{-1} H_u^I, \quad \nu(t_F) = -(1-\alpha) F\left(x^I(t_F)\right), \\
\psi(t) = H_x^I - H_{xu}^I N^{-1} H_u^I, \quad \psi(t_F) = -(1-\alpha) F_x(x^I(t_F)), \\
d\tilde{u} = -N^{-1}\left(H_u^I + H_{ux}^I dx\right), \\
H = \psi^{\mathrm{T}}(t+1) f(t, x, u) - x^0(t) - \frac{1}{2}|du|^2, \quad N = H_{uu}^I.
\end{gathered} \tag{7}
$$

It is easy to see that the first order algorithm differs fundamentally from the gradient algorithm. The variation $d\tilde{u}$ still depends on dx. Thus, as in the case of the second order algorithm, the solution is an approximate optimal linear synthesis of control.

Remark 3 Equation for σ is the matrix equation of Riccati type which can have a singular point. A point t^* is called singular if the matrix R_{uu} changes the sign-definiteness in it. In this case the singular point can be placed at point t_I at the expense of parameter α, and the variation of control can be obtained from the modified equation[20]: $R_{uu}(t_I)du(t_I) = 0$. The latter is a system of homogeneous linear algebraic equations with degenerate matrix $R_{uu}(t_I)$, therefore, a nontrivial solution always exists.

4 Constrained-end problem

4.1 Reachable sets and its estimates

The reachable set is an important characteristic of control systems. The information on reachable set (that is, its description or estimate) gives a sufficiently complete characterization, which helps to solve various control problems, in particular optimal problems.

Definition 1 The reachable set $\mathbf{X}_R(t, t_I, \mathbf{X}_I)$ of control system (1) at a time t generated by a set $\mathbf{X}_I$ at a time t_I is the set of elements $x(t)$ that can be connected by trajectories of the system beginning at the time t_I in $\mathbf{X}_I$.

Definition 2 Any set $\mathbf{X}_R^E$ that contains the reachable set is called its external estimate.

Then the following equations hold:

$$\mathbf{X}_R(t+1) = f(t, \mathbf{X}_R(t), \mathbf{U}(t, \mathbf{X}_R(t))),\ \mathbf{X}_R(t_I) = \mathbf{X}_I, \tag{8}$$

$$\mathbf{X}_R^E(t+1) = f(t, \mathbf{X}_R^E(t), \mathbf{U}(t, \mathbf{X}_R^E(t))),\ \mathbf{X}_R^E(t_I) = \mathbf{X}_I^E. \tag{9}$$

They represent the evolution of the reachable set (8) or its estimates (9).

Let $\kappa(\xi, a)$ be a scalar function and $\mathbf{X}_R^E(t) = \{x : x_n \geqslant \kappa(\xi, a(t))\}$, where $\xi = (x_1, \ldots, x_{n-1})$, and a is a vector of parameters. To obtain $\mathbf{X}_R^E(t+1)$ following Eq. (9) solve the problem $\inf\limits_{u,x} x_n(t+1)$ under conditions

$$x(t+1) = f(t, x, u),\ u \in \mathbf{U}(t, x),\ x_n \geqslant \kappa(\xi, a(t)).$$

Let the result be $\omega(t, \xi(t+1), a(t))$. Then the description of $\mathbf{X}_R^E(t+1)$ will be $x_n \leqslant \omega(t, \xi, a(t))$. Approximate $\omega(t, \xi, a(t))$ by $\kappa(\xi, a)$ on account of a according to some appropriate criterion and take the resulting a for $a(t+1)$. Thus a relation

$$a(t+1) = g(t, a(t)),\ a(t_I) = a_I,$$

is obtained which represents the evolution of $\mathbf{X}_R^E(t)$ in terms of the given function $\kappa(\xi, a)$.

Let $x(t_F) \in \mathbf{X}_F \neq \mathbb{R}^n$ in Eq. (1). It was shown in [15] that an efficient method of description of the reachable set allows one to solve strictly optimal control problems with restrictions at the right end.

Carry out a local approximation of the reachable set of the system (1) and (2). It is easily seen from Eq. (2) that x^0 is minimized when $u(t) = u^I(t)$ $(du = 0)$ for any t, thus $\left\{(x^0, x)\colon x^0 = \dfrac{1}{2} dx^{\mathrm{T}} \sigma(t) dx,\ dx = x - x^I(t)\right\}$ will be Taylor approximation of the reachable set bound with some nonnegative matrix $\sigma(t)$. It can be presented as

$$\left\{(x^0(t+1), x(t+1))\colon x^0(t+1) = \min_{du, x^0, dx(t)} \frac{1}{2}(dx(t)^{\mathrm{T}} \sigma(t) dx(t) + |du|^2)\right\}$$

under the condition

$$dx(t+1) = A(t)dx(t) + B(t)du(t),$$

which is linearized system (1) in the vicinity of $(x^I(t), u^I(t))$:

$$A(t) = f_x(t, x^I(t), u^I(t)), \quad B(t) = f_u(t, x^I(t), u^I(t)).$$

Solve the minimization problem by the Lagrange method:

$$L = \frac{1}{2}\left(dx^{\mathrm{T}} \sigma(t) dx + |du|^2\right) + \lambda^{\mathrm{T}}\left(dx(t+1) - A(t)dx(t) - B(t)du(t)\right),$$

$$L_{dx} = \sigma(t)dx - A^{\mathrm{T}}(t)\lambda = 0, \quad L_{du} = du - B^{\mathrm{T}}\lambda = 0,$$

$$dx(t+1) = A(t)\sigma^{-1}(t)A^{\mathrm{T}}(t)\lambda - B(t)B^{\mathrm{T}}(t)\lambda,$$

$$\lambda = Q^{-1}dx(t+1), \quad Q = A(t)\sigma^{-1}(t)A^{\mathrm{T}}(t) + B(t)B^{\mathrm{T}}(t),$$

$$dx = \sigma^{-1}(t)A^{\mathrm{T}}(t)Q^{-1}dx(t+1), \quad du = B^{\mathrm{T}}Q^{-1}dx(t+1),$$

$$x^0_{\min}(t+1) = \frac{1}{2}\left(dx^{\mathrm{T}}(t+1)Q^{-1}\left(A\sigma^{-1}(t)A^{\mathrm{T}} + BB^{\mathrm{T}}\right)Q^{-1}dx(t+1)\right) =$$

$$\frac{1}{2}\left(dx^{\mathrm{T}}(t+1)Q^{-1}dx(t+1)\right).$$

Thus $\sigma(t+1) = Q^{-1}$, or, in terms of $\varsigma(t) = \sigma^{-1}(t)$,

$$\varsigma(t+1) = A(t)\varsigma(t)A^{\mathrm{T}}(t) + B(t)B^{\mathrm{T}}(t). \tag{10}$$

Thus we obtain approximate reachable set that can be used to solve the minimization problem of $F_\alpha(t, x, x^0)$ with restrictions at the right end using the localization method.

4.2 Procedure of control improvement

In summary, we get the following iterative procedure.

1. Solve the linear matrix Eq. (10) for $\varsigma(t_I) = 0$.

2. Minimize the function

$$F_\alpha\left(t, x, x^0\right) = (1-\alpha)F(t,x) + \alpha x^0, \quad 0 \leqslant \alpha \leqslant 1,$$

   under the conditions

$$\left(x - x^I(t)\right) = \varsigma(t)\eta, \ \eta \in \mathbb{R}^n, \ x \in \mathbf{X}_F, \ x^0 \geqslant \frac{1}{2}\eta^{\mathrm{T}}\varsigma(t)\eta,$$

   find the minimizing point $dx_{F*}(\alpha) = \varsigma(t_F)\eta_*(\alpha)$.

3. Calculate chain

$$dx(t) = A^{-1}(t)(E^{(n)} - B(t)B^{\mathrm{T}}(t)\varsigma^{-1}(t+1))dx(t+1),$$

   backward starting at $dx(t_F) = dx_{F*}(\alpha)$ to obtain $dx_\alpha(t)$ and

$$u_\alpha(t) = u^I(t) + d\tilde{u}\left(t, dx_\alpha(t+1)\right).$$

4. Calculate forward chain (1) under $u_\alpha(t)$ starting from (t_I, x_I) to obtain $m_\alpha = (x_\alpha(t), u_\alpha(t))$ and $\Delta(\alpha) = I(m^{\mathrm{I}}) - I(m_\alpha)$.

5. Repeat steps 2-4 for different α until $(x_\alpha(t_F) \in \mathbf{X}_F)$–condition holds with given accuracy, and $\Delta(\alpha)$ is maximized at some α_*; take m_{α_*} for m^{II}.

6. Take m^{II} for m^I and start the next iteration.

The iterative process is over when $\alpha_* \approx 0$ and $m^I \approx m^{II}$ with required accuracy.

5 Examples

5.1 Parallel reaction problem

This problem was initially solved in [21] and later e.g. in [22] and [23]. Consider a tubular reactor where the following chemical reactions take place:

$$A \xrightarrow{k_1} B, \quad A \xrightarrow{k_2} C.$$

The reaction constants follow Arrhenius kinetics. We consider a discrete version of the model:

$$x_1(t+h) = x_1(t) - h\left(u(t) + 0.5u^2(t)\right)x_1(t),$$

$$x_2(t+h) = x_2(t) + hu(t)x_1(t),$$

$$x(0) = (1,\ 0)^{\mathrm{T}}, \quad h = 0.25.$$

Here, $x_1(t) = C_A/C_{Af}$ denotes the dimensionless concentration of A, $x_2(t) = C_B/C_{Bf}$ is the dimensionless concentration of B. The control $u(t) = k_1 L/v$ is restricted (L is the reactor length, v is the plug flow velocity): $0 \leqslant u(t) \leqslant 5$. The aim of the optimization is to maximize the amount of the product B until the final time: $I = x_2(1) \to \max$.

The initial approximation for the improvement procedure is $u^I = 2$ and at that $I = 0.5$. In the first step, the above first-order algorithm was applied (Remark 2). The convergence slowed after 3 iterations. Then, in the second step, the second-order algorithm from Subsection was chosen and remained active up to 2-th iteration. The value of the functional was improved to $I = 0.591841617$ that is better than the value $I = 0.5729$ reported in [24] and $I = 0.572939$ obtained in [25]. This can be explained, first, by the fact that Eq. (6) contains the matrices of second partial derivatives, and, second, $d\tilde{u}$ is a linear control synthesis. The resulting optimal control is $u = \{1.23;\ 1.22;\ 1.35;\ 1.58\}$. For the comparison only the first-order algorithm was applied with the same initial approximation. It found the optimum after 15 iterations.

5.2 Optimal helicopter maneuvering

Consider a helicopter maneuvering problem that minimizes the transferring time from one point A on the earth surface to another point B with prescribed local altitude along transfer trajectory above the surface. It is assumed that it's possible to control the velocity instantly within restrictions that depend on engine power and local directional relief slope angle.

We consider discrete motion model that was obtained through discretization of the corresponding differential equation with sufficiently small step (1 s):

$$r(t+1) = r(t) + v(\psi), r(0) = r_A,\ r(t_F) = r_B,\ I = t_F,$$

where $r = (r_1, r_2)$ is the horizontal projection of a spatial geometric point and $v = (v_1, v_2)$ is the horizontal projection of the flight velocity, $v_1 = \nu\cos\psi$, $v_2 = \nu\sin\psi$, ψ is yaw angle (the control), $\nu(\psi)$ is the maximum velocity in the direction of ψ.

The calculations were carried out for the following hypothetical data:

$$r_A = (0,0), \quad r_B = (5\,\mathrm{km}, 3\,\mathrm{km}), \quad \nu = \frac{50}{1 - 0.2\cos\left(\psi - \dfrac{\pi}{3}\right)}\frac{\mathrm{m}}{\mathrm{s}}.$$

This problem was solved with the help of the control improvement algorithm described in Subsection . To choose the initial approximation it was assumed that the helicopter flies straight through the point C, $r_C = (3$ km, 2 km); flight time is t_{ACB} = 123 s. The optimum was reached in 3 iterations and almost coincides with the evident optimal solution, that is straight flight from A to B with the maximal velocity and minimal transfer time $t_{\min}$ = 108 s.

6 Conclusion

In this paper, we introduce iterative procedures for improvement and optimization of discrete-time control systems. The algorithms are developed for both free-end and constrained-end problems based on such nontraditional approaches as the extension principle, localization principle, and approximation of reachable set. It is necessary to note that the proposed algorithms give the solution in the form of approximate linear synthesis of optimal control. The effectiveness of the proposed algorithms is demonstrated through two examples, which justifies future theoretical and experimental research of their properties.

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