

Periodic solutions of a neutral impulsive differential equation

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Abstract: In this paper, we consider a neutral impulsive differential equation. An impulsive predator-prey model with non-monotonic functional response is investigated. Some novel sufficient conditions are obtained for the nonexistence of periodic solutions and the global existence of at least one or two positive periodic solutions. Our method is based on Mawhin's coincidence degree and novel estimation techniques for the priori bounds of unknown solutions. An application is presented to illustrate the feasibility and effectiveness of our main results.

Key words: predator-prey model; periodic solution; neutral; impulsive

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一类带有脉冲的中立时滞型微分方程的周期解

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摘 要: 主要考虑了一类带有脉冲的中立时滞型微分方程. 研究了一类带有非单调函数响应的脉冲捕食模型, 得出该模型不存在周期解和存在至少一个或者两个全局周期解的充分条件. 该方法是基于 Mawhin 重合度和对未知解采用新颖的先验估计技术. 实例表明此方法是可行的也是有效的.

关键词: 捕食模型; 周期解; 中立型; 脉冲

1 Introduction

The predator-prey model plays a great role in the mathematical ecology. According to the functional response of predator to prey density and its different role in modeling, Holling^[1] proposed three types of monotonic functional

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responses $g(x) = x, \frac{x}{a+x}, \frac{x^2}{a+x^2}$ and a nonmonotonic response, so-called Holling type IV functional response $g(x) = \frac{x}{a+x+\frac{x^2}{b}}$. The multiple bifurcations of a predator-prey system with another kind of nonmonotonic functional response $g(x) = xe^{-\beta x}$ were investigated by Xiao and Ruan^[2].

To incorporate the periodicity of the environment (e.g. food supplies, seasonal effects of weather, mating habits, etc.) in many biological and ecological dynamic systems, it is necessary to consider periodicity of the parameters in the models. For examples, the authors of [3–5] have considered the predator-prey model with Holling type IV functional response by assuming a periodic environment. Recently, Xing and Xia^[6] studied the nonexistence of periodic solutions and the global existence of at least one or two periodic solutions of the following system with a general nonmonotonic response:

$$\begin{cases} \dot{x}(t) = x(t)[r(t) - a(t)x(t)] - b(t)g(x(t))y(t), \\ \dot{y}(t) = y(t)[-d(t) + c(t)g(x(t))], \end{cases} \tag{1}$$

where $g(x(t))$ is the function response, it is assumed that $g : [0, \infty) \rightarrow [0, \infty)$ is continuous and satisfies the non-monotonic condition:

- (1) $g(0) = 0$;
- (2) there exists a constant $M > 0$, such that $(x - M)g'(x) < 0$ for $x \neq M$.

It is easy to see that g is increasing on $[0, M)$, decreasing on $[M, \infty)$ and $g(x) \leq g(M)$ for $x \geq 0$. For examples

$$g(x) = \frac{x}{a^2 + x^2}, \quad g(x) = \frac{x}{\frac{x^2}{a} + x + b}, \quad g(x) = xe^{-\beta x},$$

where M, β, a, b are constants. However, there is no published paper considering the neutral impulsive system with nonmonotonic responses. Motivated by [4, 7–8], in this paper, we will consider a neutral delayed predator-prey system

$$\begin{cases} \dot{x} = x(t)[r(t) - a(t)x(t - \sigma_1(t)) - \rho(t)\dot{x}(t - \sigma_2(t))] - b(t)g(x(t))y(t), \\ \dot{y} = y(t)[-d(t) + c(t)g(x(t))], \\ \left. \begin{aligned} x(t_k^+) - x(t_k) &= \theta_{1k}x(t_k), \\ y(t_k^+) - y(t_k) &= \theta_{2k}y(t_k), \end{aligned} \right\} t = t_k, k = 1, 2, \dots, \end{cases} \tag{2}$$

with initial conditions

$$\begin{aligned} x(t) &= \varphi(t), \quad \dot{x}(t) = \dot{\varphi}(t), \quad \varphi \in C([- \sigma, 0], [0, \infty)_\phi) \cap C^1([- \sigma, 0], [0, \infty)), \quad \varphi(0) > 0, \\ y(t) &= \psi(t), \quad \dot{y}(t) = \dot{\psi}(t), \quad \psi \in C([- \sigma, 0], [0, \infty)_\phi) \cap C^1([- \sigma, 0], [0, \infty)), \quad \psi(0) > 0, \end{aligned} \tag{3}$$

where $x(t)$ and $y(t)$ represent prey and predator densities at time t , respectively. $r(t), \rho(t), a(t), b(t), c(t), d(t) \in C(\mathbb{R}, [0, +\infty)), \sigma_1(t) \in C^1(\mathbb{R}, [0, +\infty)), \sigma_2(t) \in C^2(\mathbb{R}, [0, +\infty))$ are all non-negative periodic continuous functions with ω -periodic, $\theta_{ik} > -1, i = 1, 2, k \in N^+, \sigma = \max_{t \in [0, \omega]} \{\sigma_1(t), \sigma_2(t)\}$.

The purpose of this paper is to obtain new sufficient conditions for the nonexistence of periodic solutions and the global existence of at least one or two positive periodic solutions. It should be noted that the method to obtain the global existence of at least one or two positive periodic solutions is much different from the previous work^[6] due to the introduction of the neutral term $\dot{x}(t - \sigma_2(t))$ in system (2). As usual in [6], to obtain the priori bounds of unknown solutions to the operator equation $Lz = \lambda Nz$, we need boundness of $\int_0^\omega |\dot{p}(t)| dt$. If we were to use the same ideas in [6], it is easy to obtain that

$$\int_0^\omega \left| \frac{d}{dt} p(t) \right| dt \leq (\widehat{r} + \bar{r})\omega.$$

However, due to the neutral term $\dot{x}(t - \sigma_2(t))$, it should be

$$\int_0^\omega \left| \frac{d}{dt} [p(t) + \lambda E_1(t)e^{p(t-\sigma_2(t))}] \right| dt \leq \dots \leq (\widehat{r} + \bar{r})\omega. \tag{4}$$

To obtain $\int_0^\omega |\dot{p}(t)| dt$,

$$\int_0^\omega |\dot{p}(t)| dt \leq (\widehat{r} + \bar{r})\omega + \int_0^\omega |\delta e^{p(t-\sigma_2(t))} \dot{p}(t - \sigma_2(t))| dt. \tag{5}$$

On the right side of (5), there is a term $\int_0^\omega |\delta e^{p(t-\sigma_2(t))} \dot{p}(t - \sigma_2(t))| dt$. So how to deal with this term, this is a difficulty we should overcome. The method used in the previous work [6] does not work. Thus, some novel arguments should be employed. To see how this problem is handled, the reader can refer to (16)–(29) in Section .

2 Preliminaries

Definition 1 $(x(t), y(t))^T \in C([-\sigma, \infty), (0, +\infty), (0, +\infty))$ is a solution of system (2) with initial condition (3) on $[-\sigma, +\infty)$ if

- (i) $x(t)$ and $y(t)$ are absolutely continuous on each interval $(0, t_1]$ and $(t_k, t_{k+1}]$, $k \in N^+$;
- (ii) $\forall t_k, k \in N^+, \exists (x(t_k^+), y(t_k^+))^T, (x(t_k^-), y(t_k^-))^T$, and $(x(t_k^-), y(t_k^-))^T = (x(t_k), y(t_k))^T$;
- (iii) for almost everywhere in $[0, +\infty) \setminus \{t_k\}$, $(x(t), y(t))^T$ satisfies system (2), and $x(t_k^+) - x(t_k) = \theta_{1k}x(t_k)$ and $y(t_k^+) - y(t_k) = \theta_{2k}y(t_k)$, at $t = t_k, k \in N^+$.

Throughout this paper, we make the following assumptions:

- (H₁) $\{t_i\}_{i=1,2,3,\dots}$ are fixed points which satisfy $0 < t_1 < t_2 < \dots < t_k < \dots$ and $\lim_{k \rightarrow \infty} t_k = +\infty$;
- (H₂) $\{\theta_{ik}\}$ are ω -periodic continuous functions such that $\theta_{ik} > -1$ and $\prod_{0 < t_k < t} (1 + \theta_{ik}), i = 1, 2$.

With the assumption (H₁) and (H₂), we consider the following system:

$$\begin{cases} \dot{u}(t) = u(t) \left[r(t) - A(t)u(t - \sigma_1(t)) - \delta(t)\dot{u}(t - \sigma_2(t)) - B(t)\frac{h(u(t))}{u(t)}v(t) \right], \\ \dot{v}(t) = v(t)[-d(t) + c(t)h(u(t))], \end{cases} \tag{6}$$

with the initial conditions

$$\begin{aligned} u(t) &= \varphi(t), \quad \dot{u}(t) = \dot{\varphi}(t), \quad \varphi \in C([-\sigma, 0], [0, \infty)) \cap C^1([-\sigma, 0], [0, \infty)), \quad \varphi(0) > 0, \\ v(t) &= \psi(t), \quad \dot{v}(t) = \dot{\psi}(t), \quad \psi \in C([-\sigma, 0], [0, \infty)) \cap C^1([-\sigma, 0], [0, \infty)), \quad \psi(0) > 0, \end{aligned} \tag{7}$$

where

$$\begin{aligned} A(t) &= a(t) \prod_{0 < t_k < t - \sigma_1(t)} (1 + \theta_{1k}), \quad \delta(t) = \rho(t) \prod_{0 < t_k < t - \sigma_2(t)} (1 + \theta_{1k}), \\ B(t) &= b(t) \frac{\prod_{0 < t_k < t} (1 + \theta_{2k})}{\prod_{0 < t_k < t} (1 + \theta_{1k})}, \quad g\left(\prod_{0 < t_k < t} (1 + \theta_{1k})u(t)\right) =: h(u(t)). \end{aligned} \tag{8}$$

Lemma 1 Assume that (H₁) and (H₂) are satisfied, then we have

- (i) if $(u(t), v(t))^T$ is a solution of system (6) with initial condition (7), then $(x(t), y(t))^T$ is a solution of systems (2) with initial condition (3), where

$$x(t) = \prod_{0 < t_k < t} (1 + \theta_{1k})u(t), \quad y(t) = \prod_{0 < t_k < t} (1 + \theta_{2k})v(t);$$

(ii) if $(x(t), y(t))^T$ is a solution of systems (2) with initial condition (3), then $(u(t), v(t))^T$ is a solution of systems (6) with initial condition (7), where

$$u(t) = \prod_{0 < t_k < t} (1 + \theta_{1k})^{-1} x(t), \quad v(t) = \prod_{0 < t_k < t} (1 + \theta_{2k})^{-1} y(t).$$

Proof (i) It is not difficult to verify that

$$x(t) = \prod_{0 < t_k < t} (1 + \theta_{1k}) u(t), \quad y(t) = \prod_{0 < t_k < t} (1 + \theta_{2k}) v(t),$$

are absolutely continuous on each interval $(t_k, t_{k+1}]$. For any $t \neq t_k, k \in N^+$, we have

$$\begin{aligned} & \dot{x}(t) - x(t)[r(t) - a(t)x(t - \sigma_1(t)) - \rho(t)\dot{x}(t - \sigma_2(t)) - b(t)g(x(t))y(t)] \\ &= \prod_{0 < t_k < t} (1 + \theta_{1k}) \dot{u}(t) - \prod_{0 < t_k < t} (1 + \theta_{1k}) u(t) \left[r(t) - a(t) \prod_{0 < t_k < t - \sigma_1(t)} (1 + \theta_{1k}) u(t - \sigma_1(t)) \right. \\ & \quad \left. - \rho(t) \prod_{0 < t_k < t - \sigma_2(t)} (1 + \theta_{1k}) \dot{u}(t - \sigma_2(t)) - b(t) \frac{g\left(\prod_{0 < t_k < t} (1 + \theta_{1k}) u(t)\right)}{\prod_{0 < t_k < t} (1 + \theta_{1k}) u(t)} \prod_{0 < t_k < t} (1 + \theta_{2k}) v(t) \right] \\ &= \prod_{0 < t_k < t} (1 + \theta_{1k}) \left\{ \dot{u}(t) - u(t) \left[r(t) - a(t) \prod_{0 < t_k < t - \sigma_1(t)} (1 + \theta_{1k}) u(t - \sigma_1(t)) \right. \right. \\ & \quad \left. \left. - \rho(t) \prod_{0 < t_k < t - \sigma_2(t)} (1 + \theta_{1k}) \dot{u}(t - \sigma_2(t)) - b(t) \frac{g\left(\prod_{0 < t_k < t} (1 + \theta_{1k}) u(t)\right)}{\prod_{0 < t_k < t} (1 + \theta_{1k}) u(t)} \prod_{0 < t_k < t} (1 + \theta_{2k}) v(t) \right] \right\} \quad (9) \\ &= \prod_{0 < t_k < t} (1 + \theta_{1k}) \left\{ \dot{u}(t) - u(t) \left[r(t) - A(t)u(t - \sigma_1(t)) - \delta(t)\dot{u}(t - \sigma_2(t)) \right. \right. \\ & \quad \left. \left. - B(t) \frac{g\left(\prod_{0 < t_k < t} (1 + \theta_{1k}) u(t)\right)}{u(t)} v(t) \right] \right\} \\ &= \prod_{0 < t_k < t} (1 + \theta_{1k}) \left\{ \dot{u}(t) - u(t) \left[r(t) - A(t)u(t - \sigma_1(t)) - \delta(t)\dot{u}(t - \sigma_2(t)) - B(t) \frac{h(u(t))}{u(t)} v(t) \right] \right\} \\ &= 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \dot{y}(t) - y(t)[-d(t) + c(t)g(t)] \\ &= \prod_{0 < t_k < t} (1 + \theta_{2k}) \dot{v}(t) - \prod_{0 < t_k < t} (1 + \theta_{2k}) v(t) \left[-d(t) + c(t)g\left(\prod_{0 < t_k < t} (1 + \theta_{1k}) u(t)\right) \right] \\ &= \prod_{0 < t_k < t} (1 + \theta_{2k}) \left\{ \dot{v}(t) - v(t) \left[-d(t) + c(t)g\left(\prod_{0 < t_k < t} (1 + \theta_{1k}) u(t)\right) \right] \right\} \quad (10) \\ &= \prod_{0 < t_k < t} (1 + \theta_{2k}) \left\{ \dot{v}(t) - v(t) [-d(t) + c(t)h(u(t))] \right\} \\ &= 0. \end{aligned}$$

On the other hand, $\forall t = t_k, k \in N^+$, we have

$$x(t_k^+) = \lim_{t \rightarrow t_k^+} \prod_{0 < t_j < t} (1 + \theta_{1j}) u(t) = \prod_{0 < t_j \leq t_k} (1 + \theta_{1j}) u(t_k),$$

$$y(t_k^+) = \lim_{t \rightarrow t_k^+} \prod_{0 < t_j < t} (1 + \theta_{2j})v(t) = \prod_{0 < t_j \leq t_k} (1 + \theta_{2j})v(t_k),$$

and

$$x(t_k) = \prod_{0 < t_j < t_k} (1 + \theta_{1j})u(t_k), \quad y(t_k) = \prod_{0 < t_j < t_k} (1 + \theta_{2j})v(t_k).$$

Thus, we obtain

$$x(t_k^+) = (1 + \theta_{1k})x(t_k), \quad y(t_k^+) = (1 + \theta_{2k})y(t_k). \tag{11}$$

In view of (6)–(11), we obtain that $(x(t), y(t))^T$ is a solution of system (2).

(ii) Since

$$x(t_k) = \prod_{0 < t_j < t_k} (1 + \theta_{1j})u(t_k), \quad y(t_k) = \prod_{0 < t_j < t_k} (1 + \theta_{2j})v(t_k),$$

are absolutely continuous on each interval $(t_k, t_{k+1}]$, $\forall k \in N^+$. Due to (11), for any $k \in N^+$, we have

$$u(t_k^+) = \prod_{0 < t_j \leq t_k} (1 + \theta_{1j})^{-1}x(t_k^+) = \prod_{0 < t_j < t_k} (1 + \theta_{1j})^{-1}x(t_k) = u(t_k),$$

$$v(t_k^+) = \prod_{0 < t_j \leq t_k} (1 + \theta_{2j})^{-1}y(t_k^+) = \prod_{0 < t_j < t_k} (1 + \theta_{2j})^{-1}y(t_k) = v(t_k),$$

and

$$u(t_k^-) = \prod_{0 < t_j \leq t_{k-1}} (1 + \theta_{1j})^{-1}x(t_k^-) = u(t_k),$$

$$v(t_k^-) = \prod_{0 < t_j \leq t_{k-1}} (1 + \theta_{2j})^{-1}y(t_k^-) = v(t_k).$$

The above equalities imply that $u(t)$ and $v(t)$ are continuous on $[-\sigma, \infty)$. It is easy to see that $u(t)$ and $v(t)$ are absolutely continuous on $[-\sigma, \infty)$. Similarly, we can prove that

$$u(t) = \prod_{0 < t_k < t} (1 + \theta_{1k})^{-1}x(t), \quad v(t) = \prod_{0 < t_k < t} (1 + \theta_{2k})^{-1}y(t),$$

are solutions of system (6).

From Lemma 1, we see that if we want to prove the existence of positive periodic solutions of system (2), it is sufficient to prove the existence of positive periodic solutions of system (6).

3 Nonexistence

Firstly, we shall give a necessary condition for the existence of periodic positive solutions of system (2) with initial condition (3). Notice that $h(u(t)) = g\left(\prod_{0 < t_k < t} (1 + \theta_{1k})u(t)\right)$. Thus, there exists \bar{M} such that $h(\bar{M}) = \max h(u) = \max g(x) = g(M)$.

Theorem 1 If systems (2) and (3) have a positive ω -periodic solution, then $h(\bar{M}) \geq D$.

Proof Make the change of variables

$$u(t) = e^{p(t)}, \quad v(t) = e^{q(t)},$$

then, system (6) becomes

$$\begin{cases} \dot{p}(t) = r(t) - A(t)e^{p(t-\sigma_1(t))} - \delta(t)e^{p(t-\sigma_2(t))}\dot{p}(t - \sigma_2(t)) - B(t)h(e^{p(t)})e^{q(t)-p(t)}, \\ \dot{q}(t) = -d(t) + c(t)h(e^{p(t)}). \end{cases} \tag{12}$$

Thus, system (6) is equivalent to system (12) on \mathbb{R}_+^2 . Assume that system (12) has ω -periodic solution $(p(t), q(t))^T$, i.e., $p(t + \omega) = p(t)$, $q(t + \omega) = q(t)$. Integrating the second equation of (12) on the interval $[0, \omega]$, we have

$$q(\omega) - q(0) = \int_0^\omega -d(t) + c(t)h(e^{p(t)})dt,$$

or

$$0 = -\bar{d}\omega + \int_0^\omega c(t)h(e^{p(t)})dt.$$

Since $h(\bar{M})$ is the maximum of $h(u)$,

$$\bar{d}\omega = \int_0^\omega c(t)h(e^{p(t)})dt \leq h(\bar{M})\bar{c}\omega,$$

which implies $h(\bar{M}) \geq \frac{\bar{d}}{\bar{c}} =: D$.

We complete the proof of Theorem 1. And we see the following theorem immediately.

Theorem 2 If $h(\bar{M}) < D$, then systems (2) and (3) have no positive ω -periodic solution.

4 Existence of one periodic solution

In this section, to guarantee the global existence of at least one positive periodic solution, we introduce the continuation theorem of Mawhin's coincidence degree theory from Gaines and Mawhin^[9].

Let X, Y be two real normed vector spaces, $L: \text{Dom}L \subset X \rightarrow Y$ be a linear mapping, and $N: X \rightarrow Y$ be a continuous mapping. We call the mapping L a Fredholm mapping of index zero if $\dim \text{Ker}L = \text{codim Im}L < \infty$ and $\text{Im}L$ is closed in Y . If L is a Fredholm mapping of index zero, there exist continuous projections $P: X \rightarrow X$, and $Q: Y \rightarrow Y$ such that $\text{Im}P = \text{Ker}L$, $\text{Ker}Q = \text{Im}L = \text{Im}(I - Q)$, then $L|_{\text{Dom}L \cap \text{Ker}P}: (I - P)X \rightarrow \text{Im}L$ is invertible. We denote the inverse of this map by K_p . Assume $\Omega \subseteq X$ is an open set, we say that the mapping N is L -compact on $\bar{\Omega}$, if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N: \bar{\Omega} \rightarrow X$ is compact. As $\text{Im}Q$ is isomorphic to $\text{Ker}L$, there exists an isomorphism $J: \text{Im}Q \rightarrow \text{Ker}L$.

Lemma 2^[10-11] If $\sigma_i \in C^1(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, $\dot{\sigma}_i(t) < 1$, $\sigma_i(t + \omega) = \sigma_i$, $\forall t \in [0, \omega]$, then $\phi(t) = t - \sigma_i(t)$ has a unique inverse function $\phi^{-1}(t) \in C(\mathbb{R}, \mathbb{R})$, and $\forall s \in [0, \omega]$, $\phi^{-1}(s + \omega) = \phi^{-1}(s) + \omega$.

For convenience, we introduce the following notations. If $f(t)$ is an ω -periodic continuous function,

$$\begin{aligned} \bar{f} &= \frac{1}{\omega} \int_0^\omega f(t)dt, \quad \hat{f} = \frac{1}{\omega} \int_0^\omega |f(t)|dt, \quad f^L = \min_{t \in [0, \omega]} f(t), \quad f^U = \max_{t \in [0, \omega]} f(t), \\ W_1 &\equiv \ln \left(\frac{2r^U}{E^L} \right) + \frac{2r^U(1 - (\dot{\sigma}_2)^L)}{E^L} + (\hat{r} + \bar{r})\omega, \quad W_2 \equiv \frac{(\hat{r} + \bar{r})\omega}{1 - \delta e^{W_1}}, \quad G_4 = \ln M_2 + W_2, \\ E(t) &= \frac{A(\phi^{(-1)}(t))}{1 - \dot{\sigma}_1(\phi^{(-1)}(t))}, \quad E_1 = \frac{\delta}{1 - \dot{\sigma}_2(t)}. \end{aligned}$$

Lemma 3 (Continuation theorem^[9]) Let L be a Fredholm mapping of index zero and N be L -compact on $\bar{\Omega}$. Assume

- (a) for every $\lambda \in (0, 1)$, inequality $Lz \neq \lambda Nz$ implies $z \in \partial\Omega$;
- (b) $QNz \neq 0$ for each $z \in \partial\Omega \cap \text{Ker}L$ and $\deg\{JQN, \Omega \cap \text{Ker}L, 0\} \neq 0$.

Then, the operator equation $Lz = Nz$ has at least one solution in $\text{Dom}L \cap \bar{\Omega}$.

Theorem 3 In addition to (H₁) and (H₂), further suppose that

(H₃) $h(\bar{M}) = D$;

(H₄) $1 - \delta e^{W_1} > 0, \bar{r} > \bar{A}e^{W_1};$

(H₅) $\delta(t) = \delta$ is a constant, $\dot{\sigma}_1(t) < 1, \dot{\sigma}_2(t) < 1, \ddot{\sigma}_2(t) = 0$ for $t \in \mathbb{R}$.

Then, system (6) with initial value condition (7) has at least one positive ω -periodic solution.

Proof Take

$$X = Y = \{z(t) = (p(t), q(t))^T \in C(\mathbb{R}, \mathbb{R}^2) : z(t + \omega) = z(t)\},$$

and define

$$\|z\| = \max_{t \in [0, \omega]} |p(t)| + \max_{t \in [0, \omega]} |q(t)|, z = (p, q)^T \in X \text{ or } Y,$$

where $|\cdot|$ denotes the Euclidean norm, X and Y are Banach spaces with the norm $\|\cdot\|$. Now we define the operators L, P and Q :

$$L : \text{Dom } L \cap X \rightarrow Y, \quad Lz = \frac{dz}{dt}, \quad P = \frac{1}{\omega} \int_0^\omega u(t)dt, \quad Q = \frac{1}{\omega} \int_0^\omega v(t)dt$$

where $\text{Dom } L = \{z | z \in X : z(t) = [u(t), v(t)]^T \in C^1(\mathbb{R}, \mathbb{R}^2)\}$, $L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{du(t)}{dt} \\ \frac{dv(t)}{dt} \end{pmatrix}$. Define $N : X \rightarrow Y$ as follows

$$N \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r(t) - A(t)e^{p(t-\sigma_1(t))} - \delta(t)e^{p(t-\sigma_2(t))}\dot{p}(t - \sigma_2(t)) - B(t)\frac{h(u(t))}{u(t)}v(t) \\ -d(t) + c(t)h(u(t)) \end{pmatrix}.$$

It is easy to see that

$$\text{Ker } L = \{z \in X : z = C^0, C^0 \in \mathbb{R}^2\},$$

$$\dim \text{Ker } L = \text{codim } \text{Im } L,$$

and

$$\text{Im } L = \left\{ z \in Y : \int_0^\omega z(t)dt = 0 \right\}$$

is closed in Y . P and Q are continuous projectors satisfy

$$\text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L = \text{Im}(I - Q).$$

Then, L is a Fredholm operator of index zero, and it has a unique inverse $K_p: \text{Im } L \rightarrow \text{Dom } L \cap \text{Ker } P$,

$$K_p(z) = \int_0^t z(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s)dsdt = \int_\omega^t z(s)ds + \frac{1}{\omega} \int_0^\omega sz(s)ds.$$

Then, we have

$$QNz = \begin{pmatrix} \frac{1}{\omega} \int_0^\omega [r(t) - A(t)e^{p(t-\sigma_1(t))} - \delta(t)e^{p(t-\sigma_2(t))}\dot{p}(t - \sigma_2(t)) - B(t)h(e^{p(t)})e^{q(t)-p(t)}]dt \\ \frac{1}{\omega} \int_0^\omega [-d(t) + c(t)h(e^{p(t)})]dt \end{pmatrix} \\ = \begin{pmatrix} \frac{1}{\omega} \int_0^\omega [r(t) - A(t)e^{p(t-\sigma_1(t))} - \dot{E}_1(t)e^{p(t-\sigma_2(t))} - B(t)h(e^{p(t)})e^{q(t)-p(t)}]dt \\ \frac{1}{\omega} \int_0^\omega [-d(t) + c(t)h(e^{p(t)})]dt \end{pmatrix},$$

where $E_1(t) = \frac{\delta}{1 - \dot{\sigma}_2(t)}$. By the condition (H₅), we see $\dot{E}_1(t) = 0$,

$$\begin{aligned} & \frac{1}{\omega} \int_0^\omega \delta(t) e^{p(t-\sigma_2(t))} \dot{p}(t-\sigma_2(t)) dt \\ &= \frac{1}{\omega} (E_1(t) e^{p(t-\sigma_2(t))} |_0^\omega) - \frac{1}{\omega} \int_0^\omega e^{p(t-\sigma_2(t))} d(E_1(t)) \\ &= \frac{1}{\omega} (E_1(t) e^{p(t-\sigma_2(t))} |_0^\omega). \end{aligned}$$

Now, we prove that $E_1(t) e^{p(t-\sigma_2(t))} |_0^\omega = 0$. In fact, since $\sigma_2(t)$ is an ω -periodic continuous function, $\sigma_2(t + \omega) = \sigma_2(t)$, $E_1(t)$ and $p(t)$ are ω -periodic continuous functions too. Thus,

$$e^{p(t+\omega-\sigma_2(t+\omega))} = e^{p(t+\omega-\sigma_2(t+\omega))} = e^{p(t+\omega-\sigma_2(t))} = e^{p(t-\sigma_2(t))},$$

which means $\dot{E}_1(t) e^{p(t-\sigma_2(t))} = 0$. So

$$QNz = \begin{pmatrix} \frac{1}{\omega} \int_0^\omega [r(t) - A(t) e^{p(t-\sigma_1(t))} - B(t) h(e^{p(t)}) e^{q(t)-p(t)}] dt \\ \frac{1}{\omega} \int_0^\omega [-d(t) + c(t) h(e^{p(t)})] dt \end{pmatrix},$$

and

$$\begin{aligned} K_p(I-Q)Nz &= \begin{pmatrix} \int_\omega^t [r(s) - A(s) e^{p(s-\sigma_1(s))} - B(t) h(e^{p(t)}) e^{q(t)-p(t)}] ds \\ \int_\omega^t [-d(s) + c(t) h(e^{p(t)})] ds \end{pmatrix} \\ &+ \begin{pmatrix} \int_0^\omega s [r(s) - A(s) e^{p(s-\sigma_1(s))} - E_1(s) e^{p(s-\sigma_2(s))} - B(t) h(e^{p(t)}) e^{q(t)-p(t)}] ds \\ \int_0^\omega s [-d(s) + c(t) h(e^{p(t)})] ds \end{pmatrix} \\ &+ \left(\frac{1}{2} - \frac{t}{\omega} \right) \begin{pmatrix} \int_0^\omega [r(s) - A(s) e^{p(s-\sigma_1(s))} - E_1(s) e^{p(s-\sigma_2(s))} - B(t) h(e^{p(t)}) e^{q(t)-p(t)}] ds \\ \int_0^\omega [-d(s) + c(t) h(e^{p(t)})] ds \end{pmatrix}. \end{aligned}$$

Obviously, QN and $K_p(I-Q)N$ are both continuous. Using the Arzela-Ascoli theorem, it is not difficult to prove that $K_p(I-Q)N(\bar{\Omega})$ is a compact operator and $QN(\bar{\Omega})$ is bounded for any open set $\Omega \subset X$. Thus, for any open bounded set $\Omega \subset X$, $N \in \Omega$ is L -compact.

Now we need to find an appropriate open bounded subset Ω in order to apply Lemma 3. Corresponding to the operator equation $Lz = \lambda Nz$, $\lambda \in (0, 1)$, we have the following system

$$\begin{cases} \dot{p}(t) = \lambda [r(t) - A(t) e^{p(t-\sigma_1(t))} - \delta(t) e^{p(t-\sigma_2(t))} \dot{p}(t-\sigma_2(t)) - B(t) h(e^{p(t)}) e^{q(t)-p(t)}], \\ \dot{q}(t) = \lambda [-d(t) + c(t) h(e^{p(t)})]. \end{cases} \quad (13)$$

Assume that system (13) has a solution $(p(t), q(t))^T \in X$ for a certain $\lambda \in (0, 1)$. An integration of system (13) over $[0, \omega]$ leads to

$$\int_0^\omega [A(t) e^{p(t-\sigma_1(t))} + B(t) h(e^{p(t)}) e^{q(t)-p(t)}] dt = \bar{r}\omega, \quad (14)$$

and

$$\int_0^\omega [c(t) h(e^{p(t)})] dt = \bar{d}\omega. \quad (15)$$

Therefore, we have

$$\begin{aligned} & \int_0^\omega \left| \frac{d}{dt} [p(t) + \lambda E_1(t)e^{p(t-\sigma_2(t))}] \right| dt \\ &= \lambda \int_0^\omega |r(t) - A(t)e^{p(t-\sigma_1(t))} - B(t)h(e^{p(t)})e^{q(t)-p(t)}| dt \\ &\leq \int_0^\omega |r(t)| dt + \int_0^\omega [A(t)e^{(t-\sigma_1(t))} + B(t)h(e^{p(t)})e^{q(t)-p(t)}] dt \\ &= (\widehat{r} + \bar{r})\omega. \end{aligned} \tag{16}$$

Similarly, from the second equation of (13), and (15), we get

$$\int_0^\omega |\dot{q}(t)| dt \leq (\widehat{d} + \bar{d})\omega. \tag{17}$$

Note that

$$h(\bar{M}) = D = \frac{\bar{d}}{\bar{c}}. \tag{18}$$

Then, there exists $\xi_1 \in [0, \omega]$ such that

$$p(\xi_1) = \ln \bar{M}. \tag{19}$$

Let $t - \sigma_1 = s$. By Lemma 2 and (14), we have

$$\int_0^\omega A(t)e^{p(t-\sigma_1(t))} dt = \int_{-\sigma_1(0)}^{\omega-\sigma_2(t)} \frac{A(\phi^{-1}(t))e^{p(t)}}{1 - \dot{\sigma}_1(\phi^{-1}(t))} dt = \int_0^\omega E(t)e^{p(t)} dt < \bar{r}\omega. \tag{20}$$

By the mean value theorem in calculus, there exists a number $\xi_2 \in [0, \omega]$ such that

$$\int_0^\omega E(t)e^{p(t)} dt = E(\xi_2) \int_0^\omega e^{p(t)} dt = E(\xi_2) \int_0^\omega e^{p(t-\sigma_2(t))} (1 - \dot{\sigma}_2(t)) dt < \bar{r}\omega. \tag{21}$$

For some number $\xi_3 \in [0, \omega]$ such that

$$E(\xi_2) \int_0^\omega \frac{e^{p(t-\sigma_2(t))}}{1 - \dot{\sigma}_2(t)} dt = E(\xi_2)(1 - \dot{\sigma}_2(\xi_3)) \int_0^\omega e^{p(t-\sigma_2(t))} dt < \bar{r}\omega. \tag{22}$$

Then, for some $\xi \in [0, \omega]$,

$$E(\xi_2)e^{u(\xi)} + E(\xi_2)(1 - \dot{\sigma}_2(\xi_3))e^{p(\xi-\sigma_2(\xi))} < 2r(\xi). \tag{23}$$

By $\int_0^\omega r(t) > 0$, so $r^U > 0$; and from condition (H₅) $1 - \dot{\sigma}_2(t) > 0$, we have

$$p(\xi) < \ln \left(\frac{2r^U}{E^L} \right), \tag{24}$$

$$p(\xi - \sigma_2(\xi)) < \ln \left(\frac{2r^U}{E^L(1 - (\dot{\sigma}_2)^L)} \right), \tag{25}$$

and

$$\begin{aligned} p(t) + \lambda E_1(t)e^{p(t-\sigma_2(t))} &\leq p(\xi) + \lambda E_1(\xi)e^{p(\xi-\sigma_2(\xi))} \\ &+ \int_0^\omega \left| \frac{d}{dt} [p(t) + \lambda E_1(t)e^{p(t-\sigma_2(t))}] \right| dt \\ &< \ln \left(\frac{2r^U}{E^L} \right) + E_1^U \frac{2r^U}{E^L(1 - (\dot{\sigma}_2)^L)} + (\widehat{r} + \bar{r})\omega \\ &=: W_1. \end{aligned} \tag{26}$$

In view of $E_1(t)e^{p(t-\sigma_2(t))} > 0$, we have

$$p(t) < W_1. \quad (27)$$

From (13), (14), (16) and (27), we have

$$\begin{aligned} \int_0^\omega |\dot{p}(t)|dt &\leq \int_0^\omega |r(t)|dt + \int_0^\omega [A(t)e^{p(t-\sigma_1(t))} + B(t)h(e^{p(t)})e^{q(t)-p(t)}]dt \\ &\quad + \int_0^\omega |\delta e^{p(t-\sigma_2(t))}\dot{p}(t-\sigma_2(t))|dt \\ &\leq (\widehat{r} + \bar{r})\omega + \delta e^{W_1} \int_0^\omega |\dot{p}(t)|dt. \end{aligned} \quad (28)$$

In view of $(H_4) 1 - \delta e^{W_1} > 0$, we have

$$\int_0^\omega |\dot{p}(t)|dt \leq \frac{(\widehat{r} + \bar{r})\omega}{1 - \delta e^{W_1}} =: W_2. \quad (29)$$

From (19) and (29),

$$p(t) \geq p(\xi_1) - \int_0^\omega |\dot{p}(t)|dt > \ln \bar{M} - W_2 =: W_3, \quad (30)$$

and

$$p(t) \leq p(\xi_1) + \int_0^\omega |\dot{p}(t)|dt < \ln \bar{M} + W_2 =: W_4. \quad (31)$$

Since $(p(t), q(t))^T \in X$, there exist $\xi^*, \xi_* \in [0, \omega]$ such that

$$q(\xi^*) = \max_{t \in [0, \omega]} q(t), \quad q(\xi_*) = \min_{t \in [0, \omega]} q(t).$$

Note that $h(\bar{M}) = D$ is the maximum of the function h , and it follows from (14), (29), (31) and condition $(H_4) \bar{r} > \bar{A}e^{W_1}$ that

$$\bar{A}\omega e^{W_1} + \bar{B}\omega D e^{q(\xi^*)-W_3} > \bar{r}\omega,$$

and

$$q(\xi^*) \geq \ln \frac{\bar{r}e^{W_3} - \bar{A}e^{W_1+W_3}}{\bar{B}D} =: W_5. \quad (32)$$

Combining (32) with (17), we have

$$q(t) \geq q(\xi^*) - \int_0^\omega |\dot{q}(t)|dt > W_5 - (\widehat{d} + \bar{d})\omega =: W_6. \quad (33)$$

Note that $e^{W_3} < \bar{M}$, $e^{W_4} > \bar{M}$. Since h is increasing on $[0, \bar{M})$ and decreasing on (\bar{M}, ∞) , we have

$$h(e^{p(t)}) \geq \min\{h(e^{W_3}), h(e^{W_4})\}, \quad p(t) \in (W_3, W_4).$$

Combining this with (17), (27), (30) and (31), we have

$$\bar{B}\omega \min\{h(e^{W_3}), h(e^{W_4})\}e^{q(\xi_*)-W_1} \leq \bar{r}\omega.$$

Thus, we get

$$q(\xi_*) \leq \ln \frac{\bar{r}e^{W_1}}{\bar{B} \min\{h(e^{W_3}), h(e^{W_4})\}} =: W_7. \quad (34)$$

Combining (34) with (17), we have

$$q(t) \leq q(\xi_*) + \int_0^\omega |\dot{q}(t)|dt < W_7 + (\widehat{d} + \bar{d})\omega =: W_8. \quad (35)$$

Let $W = \max\{|\ln \bar{M}|, |W_1|, |W_3|, |W_4|\}$ and $\bar{W} = \max\{|W_6|, |W_8|\}$. Then, from (19), (27), (30), (31), (33) and (35), we have $|p(t)| < W$, $|q(t)| < \bar{W}$, and W, \bar{W} are independent of λ . Consider the equation $QNz = 0$, within $z = (p, q)^T \in \mathbb{R}^2$, i.e.,

$$QNz = 0. \tag{36}$$

In view of (H₄), we have $\bar{r} - \bar{A}e^{W_1} > 0$, and (H₃), it is not difficult to show that (36) has a unique solution $(p^*, q^*)^T = (\ln \bar{M}, q^*)^T$. Let $C > 0$, such that $\|(p^*, q^*)^T\| = |p^*| + |q^*| < C$. Set

$$\Omega = \{z \in X : \|z\| < W + \bar{W} + C\}.$$

It is easy to see that Ω satisfies the condition of Lemma 3. When $z = (p, q)^T \in \partial\Omega \cap \text{Ker}L = \partial\Omega \cap \mathbb{R}^2$, z is a constant vector in \mathbb{R}^2 with $\|z\| = W + \bar{W} + C$ and we have $QNz \neq 0$. Moreover, we define the homomorphism $J: \text{Im}Q \rightarrow \text{Ker}L$, the identity mapping. By the assumption in Theorem 3, it is easy to prove that

$$\text{deg}\{JQN, \Omega \cap \text{Ker}L, 0\} \neq 0,$$

where $\text{deg}(\cdot)$ is the Brouwer degree^[9]. Now, we have proved that Ω satisfies all the requirements of Lemma 3. It follows that $Lz = Nz$ has at least one solution in $\text{Dom}L \cap \bar{\Omega}$.

5 Existence of two periodic solution

In this section, we consider system (12) under the assumption $h(\bar{M}) > D$. From the non-monotonic condition, if $h(\bar{M}) > D$, then the equation $h(\bar{M}) = D$ has two positive solutions M_1 and M_2 such that

$$h(M_1) = h(M_2) = D, \quad 0 < M_1 < \bar{M} < M_2.$$

Theorem 4 In addition to (H₁) and (H₂), we further suppose that

(H₅) $\delta(t) = \delta$ is a constant, $\dot{\sigma}_1(t) < 1$, $\dot{\sigma}_2(t) < 1$, $\ddot{\sigma}_2(t) = 0$ for $t \in \mathbb{R}$;

(H₆) $h(\bar{M}) > D$;

(H₇) $1 - \delta e^{W_1} > 0$, $1 - \delta \bar{M} > 0$, $\bar{r} - \bar{A}\bar{M} > 0$, $\bar{r} - \bar{A}e^{G_4} > 0$, $\ln \bar{M}_1 + W_2 \leq \ln \bar{M}$, $\ln \bar{M}_2 - W_2 \geq \ln \bar{M}$.

Then, system (6) has at least two positive ω -periodic solutions.

Proof In order to prove that system (2) has two positive ω -periodic solutions, the most important task is to search for at least two appropriate open bounded subsets Ω_1, Ω_2 in X for the application of Lemma 3. Let X, Y, L, N, P, Q be defined as in the proof of Theorem 3, and let $z = (p(t), q(t))^T \in X$ be a solution of $Lz = \lambda Nz$ for a certain $\lambda \in (0, 1)$.

Since $(p(t), q(t))^T \in X$, there exist $\eta_*, \eta^* \in [0, \omega]$ such that

$$\begin{aligned} p(\eta_*) &= \min_{t \in [0, \omega]} p(t), & q(\eta_*) &= \min_{t \in [0, \omega]} q(t), \\ p(\eta^*) &= \max_{t \in [0, \omega]} p(t), & q(\eta^*) &= \max_{t \in [0, \omega]} q(t). \end{aligned} \tag{37}$$

We claim that $p(t) \neq \ln \bar{M}$, $t \in [0, \omega]$. Otherwise, it implies that $h(\bar{M}) = D$, which contradicts (H₆). Therefore, either $p(t) \in (-\infty, \ln \bar{M})$ or $p(t) \in (\ln \bar{M}, \infty)$.

Case 1: $p(t) < \ln \bar{M}, t \in [0, \omega]$. From (14), (H₆) and the non-monotonic condition, there exists $\theta_1 \in [0, \omega]$ such that

$$\bar{d}\omega = \bar{c}\omega h(e^{p(\theta_1)}).$$

Then, we have

$$e^{p(\theta_1)} = h^{-1}(D) \in (0, \bar{M}) = M_1.$$

From (13), (14), (16) and $p(t) < \ln \bar{M}$, we have

$$\begin{aligned} \int_0^\omega |\dot{p}(t)| dt &\leq \int_0^\omega |r(t)| dt + \int_0^\omega [A(t)e^{p(t-\sigma_1(t))} + B(t)h(e^{p(t)})e^{q(t)-p(t)}] dt \\ &\quad + \int_0^\omega |\delta e^{p(t-\sigma_2(t))} \dot{p}(t - \sigma_2(t))| dt \\ &\leq (\hat{r} + \bar{r})\omega + \delta \bar{M} \int_0^\omega |\dot{p}(t)| dt. \end{aligned}$$

In view of (H₇), that is, $1 - \delta \bar{M} > 0$, we have

$$\int_0^\omega |\dot{p}(t)| dt \leq \frac{(\hat{r} + \bar{r})\omega}{1 - \delta \bar{M}} =: W_9.$$

Thus, we have

$$p(t) \leq p(\theta_1) + \int_0^\omega |\dot{p}(t)| dt < \ln M_1 + W_9 \leq \ln \bar{M}, \quad (38)$$

and

$$p(t) \geq p(\theta_1) - \int_0^\omega |\dot{p}(t)| dt > \ln M_1 - W_9 =: G_1. \quad (39)$$

It follows from (38) and (39) that

$$p(t) \in (G_1, \ln \bar{M}). \quad (40)$$

Moreover, it follows from (14), (38) and (40) that

$$\bar{B}\omega h(e^{G_1})e^{q(\eta^*) - \ln \bar{M}} \leq \bar{r}\omega.$$

That is,

$$q(\eta^*) \leq \ln \frac{\bar{r}\bar{M}}{\bar{B}h(e^{G_1})}, \quad (41)$$

which, together with (17), leads us to

$$q(t) \leq q(\eta^*) + \int_0^\omega |\dot{q}(t)| dt < \ln \frac{\bar{r}\bar{M}}{\bar{B}h(e^{G_1})} + (\hat{d} + \bar{d})\omega =: G_2. \quad (42)$$

From (H₆) and (H₇), it is easy to see that $\bar{r} - \bar{A}\bar{M} > 0$. It follows from (14), (37) and (40) that

$$\bar{A}\omega e^{\ln \bar{M}} + \bar{B}\omega h(\bar{M})e^{q(\eta^*) - G_1} \geq \bar{r}\omega.$$

That is,

$$q(\eta^*) \geq \ln \frac{(\bar{r} - \bar{A}\bar{M})G_1}{\bar{B}h(\bar{M})}, \quad (43)$$

which, combined with (17), leads us to

$$q(t) \geq q(\eta^*) - \int_0^\omega |\dot{q}(t)| dt > \ln \frac{(\bar{r} - \bar{A}\bar{M})G_1}{\bar{B}h(\bar{M})} - (\hat{d} + \bar{d})\omega =: G_3. \quad (44)$$

Therefore, from (42) and (44), we have

$$\max_{t \in [0, \omega]} |q(t)| < \max\{G_2, G_3\} =: G, \quad (45)$$

for $p(t) \in (G_1, \ln \bar{M})$.

Case 2: $p(t) > \ln \bar{M}$, $t \in [0, \omega]$. From (14), (H₆) and the non-monotonic condition, there exists $\theta_2 \in [0, \omega]$ such that

$$\bar{d}\omega = \bar{c}\omega h(e^{p(\theta_2)}).$$

Then, we have

$$e^{p(\theta_2)} = h^{-1}(D) \in (0, \bar{M}) = M_2.$$

Combining this with (28) and (H₇), we get

$$p(t) \leq p(\theta_2) + \int_0^\omega |\dot{p}(t)| dt < \ln M_2 + W_2 =: G_4, \quad (46)$$

and

$$p(t) \geq p(\theta_2) - \int_0^\omega |\dot{p}(t)| dt > \ln M_2 - W_2 \geq \ln \bar{M}. \quad (47)$$

It follows from (46) and (47) that

$$p(t) \in (\ln \bar{M}, G_4). \quad (48)$$

Since $h(x)$ is decreasing for $x \in (\bar{M}, \infty)$, it follows from (14), (37) and (48) that

$$\bar{B}\omega h(e^{G_4})e^{q(\eta^*)-G_4} \leq \bar{r}\omega.$$

We rewrite this in the following way

$$q(\eta^*) \leq \ln \frac{\bar{r}e^{G_4}}{\bar{B}h(e^{G_4})}. \quad (49)$$

Together with (17), this leads to

$$q(t) \leq q(\eta^*) + \int_0^\omega |\dot{q}(t)| dt < \ln \frac{\bar{r}e^{G_4}}{\bar{B}h(e^{G_4})} + (\hat{d} + \bar{d})\omega =: G_5. \quad (50)$$

Moreover, it follows from (14), (37) and (48), and condition (H₆) $\bar{r} - \bar{A}e^{G_4} > 0$ that

$$\bar{A}\omega e^{G_4} + \bar{B}\omega h(\bar{M})e^{q(\eta^*)-\ln \bar{M}} \geq \bar{r}\omega.$$

That is,

$$q(\eta^*) \geq \ln \frac{(\bar{r} - \bar{A}e^{G_4})\bar{M}}{\bar{B}h(\bar{M})}, \quad (51)$$

which, combined with (17), gives us

$$q(t) \geq q(\eta^*) - \int_0^\omega |\dot{q}(t)| dt > \ln \frac{(\bar{r} - \bar{A}e^{G_4})\bar{M}}{\bar{B}h(\bar{M})} - (\hat{d} + \bar{d})\omega =: G_6. \quad (52)$$

Therefore, from (50) and (52), we have

$$\max_{t \in [0, \omega]} |q(t)| < \max\{G_5, G_6\} =: \bar{G} \quad (53)$$

for $p(t) \in (\ln \bar{M}, G_4)$. Obviously, $M_1, M_2, \ln \bar{M}, G_1, G_3, G, \bar{G}$ are independent of λ . Consider the equation $Q Nz = 0$, within $z = (p, q)^T \in \mathbb{R}^2$, i.e., (36) and the non-monotonic condition (H₆), we show that (36) has two distinct solutions

$$(\bar{p}, \bar{q})^T = (\ln M_1, \bar{q})^T,$$

and

$$(\hat{p}, \hat{q})^T = (\ln M_2, \hat{q})^T.$$

Choose $C_1 > 0$ such that

$$\max(\bar{q}, \hat{q}) < C_1. \quad (54)$$

Define

$$\Omega_1 = \left\{ z = (p, q)^T \in X : p(t) \in (G_1, \ln \bar{M}), \max_{t \in [0, \omega]} |q(t)| < G + C_1 \right\},$$

$$\Omega_2 = \left\{ z = (p, q)^T \in X : p(t) \in (\ln \bar{M}, G_4), \max_{t \in [0, \omega]} |q(t)| < \bar{G} + C_1 \right\}.$$

Both Ω_1 and Ω_2 are bounded open subsets of X . It follows from the non-monotonic condition, (45) and (53) that $(\bar{p}, \bar{q})^T \in \Omega_1$, $(\hat{p}, \hat{q})^T \in \Omega_2$. In view of (45) and (53), we have $\Omega_1 \cap \Omega_2 = \emptyset$. Thus, $\Omega_i, i = 1, 2$ satisfy condition (a) of Lemma 3. Furthermore, $QNz \neq 0$ for $z \in \partial\Omega_i \cap \text{Ker}L = \partial\Omega_i \cap \mathbb{R}^2$. It is not difficult to verify that $\deg\{JQN, \Omega_i \cap \text{Ker}L, 0\} \neq 0$. Thus, Ω_i satisfy all the conditions of Lemma 3. Hence, $Lz = Nz$ has at least two ω -periodic solutions $z^* = (p^*, q^*)^T$ and $z_* = (p_*, q_*)^T$ with $z^* \in \text{Dom}L \cap \bar{\Omega}_1$ and $z_* \in \text{Dom}L \cap \bar{\Omega}_2$, and z^*, z_* are different. Therefore, $(e^{p^*}, e^{q^*})^T, (e^{p_*}, e^{q_*})^T$ are two different positive ω -periodic solutions of (6).

6 Application

In this section, we will present an application of Theorems 1, 2, 3, 4. Consider the following non-monotonic periodic system

$$\left\{ \begin{array}{l} \dot{x} = x(t) \left[6 + \cos t - (5 + \cos t)x \left(t - \frac{\sin t}{2} \right) - 10^{-10} \dot{x}(t-1) \right] - (2 + \sin t) x e^{-\beta x} y(t), \\ \dot{y} = y(t) [3 + \sin t + (10 + \cos t) x e^{-\beta x}], \\ \left. \begin{array}{l} x(t_k^+) - x(t_k) = -0.1x(t_k), \\ y(t_k^+) - y(t_k) = 0.1y(t_k), \end{array} \right\} \begin{array}{l} t \neq 2k\pi, \\ t = 2k\pi, k = 1, 2, \dots, \end{array} \end{array} \right. \quad (55)$$

where

$$g(x) = x e^{-\beta x}.$$

Note that g is a non-monotonic function. $g'(M) = 0$ implies that $M = \frac{1}{\beta}$ and $g(M) = \frac{1}{\beta e}$. Corresponding to system (8), we have

$$A(t) = a(t) \prod_{0 < t_k < t - \sigma_1(t)} (1 + \theta_{1k}), \quad \delta(t) = \rho(t) \prod_{0 < t_k < t - \sigma_2(t)} (1 + \theta_{1k}),$$

$$B(t) = b(t) \prod_{0 < t_k < t} (1 + \theta_{2k}), \quad \theta = \prod_{0 < t_k < t} (1 + \theta_{1k}) \quad (56)$$

Corresponding to system (12), we have $\sigma_1' = \frac{\cos t}{2} < 1$, $\sigma_2'' = 0$, $\bar{r} = 6$, $\bar{A} = 5\theta$, $\delta = 10^{-10}\theta$, $\bar{B} = 2\theta$, $\frac{\bar{d}}{c} = \frac{3}{10}$. Notice that $h(u(t)) = g\left(\prod_{0 < t_k < t} (1 + \theta_{1k})u(t)\right)$. Thus, there exists \bar{M} such that $h(\bar{M}) = \max h(u) = \max g(x) = \frac{1}{\beta e}$. Then, we have the following results.

Theorem 5 If $\frac{1}{\beta e} < \frac{\bar{d}}{c}$, system (55) has no positive ω -periodic solution.

Theorem 6 Suppose that

(S₁) If $\frac{1}{\beta e} = \frac{\bar{d}}{c}$;

(S₂) $W_1 > 24\pi$, $\delta e^{W_1} \ll 1$, $1 - \delta e^{W_1} > 0$, $6 > 5\theta e^{W_1}$;

(S₃) $\delta(t) = \delta$ is constant; $\dot{\sigma}_1(t) < 1$, $\dot{\sigma}_2(t) < 1$, $\ddot{\sigma}_2(t) = 0$ for $t \in \mathbb{R}$.

Then, system (55) has at least one positive ω -periodic solution.

Theorem 7 Suppose that

(S₃) $\delta(t) = \delta$ is constant; $\dot{\sigma}_1(t) < 1$, $\dot{\sigma}_2(t) < 1$, $\ddot{\sigma}_2(t) = 0$ for $t \in \mathbb{R}$;

(S₄) $\frac{1}{\beta e} > \frac{\bar{d}}{\bar{c}}$;

(S₅) $1 - \delta e^{W_1} > 0$, $1 - \delta M > 0$, $\bar{r} - \bar{A}M > 0$, $\bar{r} - \bar{A}e^{G_4} > 0$, $\ln M_1 + W_2 \leq \ln \bar{M}$, $\ln M_2 - W_2 \geq \ln \bar{M}$.

Then, system (55) has at least two positive ω -periodic solutions.

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