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On algebraic convergence of non-elementary discrete subgroups of $\mathrm{SL}(2, \mathbb{Q}_p)$

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Abstract: In the Kelinian groups, the study of the algebraic convergence of the sequence of the discrete subgroups is a very important topic, since the algebraic convergence of the sequence of the discrete subgroups can be applied to study the deformations the manifolds and the Hausdorff dimension of the limit sets of the discrete subgroups. With the rapid developments of the p-adic Lie groups and the algebraic dynamical systems, it is very important to study the topics of algebraic convergence of the p-adic discrete subgroups. In this paper, we discuss the algebraic convergence of a sequence $\{G_{n,r}\}$ of r-generator non-elementary discretesubgroups of $\mathrm{PSL}(2, \mathbb{Q}_p)$ by use of the Jorgensen inequalities in $\mathrm{PSL}(2, \mathbb{Q}_p)$ and the subgroups of $\mathrm{PSL}(2, \mathbb{Q}_p)$ acting isometrically on the hyperbolic Berkovich space. We prove that a sequence $\{G_{n,r}\}$ of r-generator non-elementary discretesubgroups of $\mathrm{PSL}(2, \mathbb{Q}_p)$ converges to a non-elementary discrete subgroup of $\mathrm{PSL}(2, \mathbb{Q}_p)$ algebraically.

Key words: p-adic Mobius maps; algebraic convergence; discrete subgroups

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$\mathrm{SL}(2, \mathbb{Q}_p)$ 中的非初等离散子群的代数收敛性

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摘 要：在 Kleinian 群中，研究离散群的代数收敛性是一个重要的问题，群列的代数收敛性与流形的形变以及极限集的 Hausdorff 维数的收敛性有密切关系．随着非阿基米德域上的李群和非阿基米德域上的动力系统的发展，讨论非阿基米德域上的离散群的代数收敛性就是一个重要的问题．讨论了 $\mathrm{PSL}(2, \mathbb{Q}_p)$ 中由 r 个元素生成的非初等离散群的代数收敛性，利用 $\mathrm{PSL}(2, \mathbb{Q}_p)$ 中关于子群的非阿基米德 Jorgensen 不等式，以及群双曲 Berkovich 空间上的双曲等距性，证明了非初等群列代数收敛到非初等群列上．

关键词：p-adic Mobius 变换；代数收敛性；离散群

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1 Introduction

Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$, where $\mathbb{Q}_p$ is the field of p-adic rational numbers. For $\mathbb{C}_p$, the projective space $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ is totally disconnected and not locally compact. The Berkovich projective space $\mathbb{P}^1_{\mathrm{Ber}}$ is a compact augmentation of $\mathbb{P}^1(\mathbb{C}_p)$ and is also a uniquely path-connected Hausdorff space which contains $\mathbb{P}^1(\mathbb{C}_p)$ as its dense subset. By Berkovich's classification theorem, points of $\mathbb{P}^1_{\mathrm{Ber}} \backslash \mathbb{P}^1(\mathbb{C}_p)$ correspond to discs $D(a, r) \subset \mathbb{C}_p$, or to nested sequences of discs. The point ζ_G corresponding to $D(0, 1)$ is called the Gauss point. The element in the projective special linear group $\mathrm{PSL}(2, \mathbb{C}_p)$ is called the p-adic Möbius map. The action of the p-adic Möbius map on $\mathbb{P}^1(\mathbb{C}_p)$ continuously extends to an isometrical map on the hyperbolic Berkovich space $\mathbb{H}_{\mathrm{Ber}} = \mathbb{P}^1_{\mathrm{Ber}} \setminus \mathbb{P}^1(\mathbb{C}_p)$. For the foundation, see [1–3], more details can be found in section 2.

Consider a sequence of subgroups $\{G_{n,i}\}$ generated by r p-adic Möbius maps, say $g_{n,1}, \cdots, g_{n,r}$. If for each $k \in \{1, \cdots, r\}$, the sequence $g_{n,k}$ converges to a p-adic Möbius maps g_k then one says that $\{G_{n,i}\}$ converges algebraically to the group G generated by $g_1, \cdots, g_r$, or G is the algebraic limit of the sequence $\{G_{n,i}\}$.

There are many discussions of algebraic convergence of Kleinian groups; see [4–6]. With the development of studying the arithmetical dynamics, it is an important topic in the theory of p-adic Möbius maps in non-archimedean spaces (see [3, 7–11]). However, it is also an open question in the arithmetical settings to study the algebraic convergence of discrete subgroups of $\mathrm{PSL}(2, \mathbb{Q}_p)$. Hence we give a clear picture of it.

Theorem 1 Let the sequence of non-elementary discrete groups $\{G_n\}$ of $r-$generators converges to the group G. Then the group G is also non-elementary discrete.

2 Some basic facts

Let $p \geqslant 2$ be a prime number. Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Denote $|\mathbb{C}_p^*|$ the valuation group of $\mathbb{C}_p$. Then every element $r \in |\mathbb{C}_p^*|$ can be expressed as $r = p^s$ with $s \in \mathbb{Q}$. The absolute value satisfies the strong triangle inequality

$$|z - y| \leqslant \max\{|z|, |y|\}$$

for $x, y \in \mathbb{C}_p$. If x, y and z are points of $\mathbb{C}_p$ with $|x - y| < |x - z|$, then $|x - z| = |y - z|$.

Given $a \in \mathbb{C}_p$, and $r > 0$, the open and closed disks with center at a with radius r are defined by

$$D(a, r)^- = \{z \in \mathbb{C}_p \mid |z - a| < r\},$$

$$D(a, r) = \{z \in \mathbb{C}_p \mid |z - a| \leqslant r\}.$$

However, by the strong triangle inequality, topologically $D(a, r)^-$ and $D(a, r)$ are closed and open, and every point in disk $D(a, r)^-$ is the center. This implies that if $x \in D(a, r)^-$, then $D(a, r)^- = D(x, r)^-$ (resp. $D(a, r) = D(x, r)$). if two disks D_1 and D_2 in $\mathbb{C}_p$ have non-empty intersection, then $D_1 \subset D_2$, or $D_2 \subset D_1$. For a set $E \subset \mathbb{C}_p$, denote $\mathrm{diam}(E) = \sup\limits_{z,w \in E} |z - w|$ the diameter of E in the non-Archimedean metric. Especially, $\mathrm{diam}(D(a, r)) = r$.

Let $\mathbb{P}^1(\mathbb{C}_p)$ be the projective space over $\mathbb{C}_p$ which can be viewed as $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$. The chordal distance on $\mathbb{P}^1(\mathbb{C}_p)$ can be defined by

$$\rho_v(z,w)=\begin{cases}\dfrac{|z-w|}{\max\{1,|z|\}\max\{1,|w|\}}, & \text{for } z,w\in\mathbb{C}_p\\ \dfrac{1}{\max\{1,|w|\}}, & \text{for } w\in\mathbb{C}_p \text{ and } z=\infty\\ 0, & \text{for } w=\infty \text{ and } z=\infty\end{cases}.$$

By the definition of the chordal distance and the strong triangle inequality, it is easy to show that if $|z|\leqslant 1, |w|\leqslant 1$, then $\rho_v(z,w)=|z-w|$, and if $|z|>1,|w|\leqslant 1$, then $\rho_v(z,w)=\dfrac{|z-w|}{|z|}=1$, and if $|z|>1,|w|>1$, then $\rho_v(z,w)=\dfrac{|z-w|}{|z||w|}=\Big|\dfrac{1}{z}-\dfrac{1}{w}\Big|$.

In [8], Kato introduced the method that was used in Kleinian groups to study the discrete subgroups of $\mathrm{PSL}(2,\mathbb{C}_p)$. He classified non-unit elements g in $\mathrm{PSL}(2,\mathbb{C}_p)$ into the following three classes:

(a) g is said to be parabolic if it has only one eigenvalue.

(b) g is said to be elliptic if it has two distinct eigenvalues λ_1 and λ_2 with $|\lambda_1|=|\lambda_2|$.

(c) g is said to be hyperbolic if it has two eigenvalues λ_1 and λ_2 with $|\lambda_1|\neq|\lambda_2|$.

For $g=(a_{ij})$ in the matrix ring $\mathrm{M}(m,\mathbb{Q}_p)$, the norm of g is defined by $\|g\|=\max\limits_{1\leqslant i\leqslant m,1\leqslant j\leqslant m}\{|a_{ij}|\}$. Obviously, $\|g\|=0$ implies that each $a_{ij}=0$. It is easy to verify that

$$\|\alpha g\|=|\alpha|\|g\|,\quad \|g+h\|\leqslant\max\{\|g\|,\|h\|\},\quad\text{and}\quad \|gh\|\leqslant\|g\|\|h\|.$$

We say that a subgroup G of $\mathrm{SL}(m,\mathbb{Q}_p)$ is discrete if there exists $\delta=\delta(G)>0$ such that each element $g\in G\setminus\{I\}$ satisfies $\|g-I\|>\delta$, where I denotes the unit element.

Obviously, a subgroup G of $\mathrm{SL}(2,\mathbb{Q}_p)$ is discrete if and only if any sequence consisting of distinct elements $g_n\in G$ is not a Cauchy sequence. Since $\|h^{-1}g_nh-h^{-1}gh\|\leqslant\|h^{-1}\|\|g_n-g\|\|h\|$, we have $\|h^{-1}g_nh-h^{-1}gh\|\longrightarrow 0$, when $g_n\longrightarrow g$, as $n\longrightarrow\infty$. This means that conjugation does not change the discreteness.

We say that G is elementary if there exists a finite $G-$orbit in $\mathbb{P}^1_{\mathrm{Ber}}$, and G acts discontinuously on $\mathbb{H}_{\mathrm{Ber}}$ if for any sequence $\{g_n\}$, the sequence $\{g_n(x)\}$ has no convergent subsequence in $\mathbb{H}_{\mathrm{Ber}}$ for any point $x\in\mathbb{H}_{\mathrm{Ber}}$ with respect to the Berkovich topology.

Lemma 1 [3] If a discrete subgroup G of $\mathrm{PSL}(2,\mathbb{C}_p)$ is elementary, then either G is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{C}_p)$, every element of which leaves $\{0,\infty\}$ invariant and is therefore of the form $z\longrightarrow az^s$, $a\neq 0, s^2=1$, or each element in G shares the same fixed point in $\mathbb{H}_{\mathrm{Ber}}$.

Lemma 2 [8] Let G be a discrete subgroup of $\mathrm{SL}(m,\mathbb{Q}_p)$. Then

(1) there is no parabolic element in G;

(2) there is no elliptic element of infinite order in G.

Lemma 3 [12] The subgroup G of $\mathrm{SL}(m,\mathbb{Q}_p)$ is discrete if and only if any cyclic subgroup of G is discrete.

Lemma 4 [11] If a discrete subgroup G of $\mathrm{SL}(2,\mathbb{Q}_p)$ contains elliptic elements only, then G is a finite group.

Lemma 5 [12] If a subgroup G of $\mathrm{SL}(2,\mathbb{C}_p)$ is discrete, then for each $g\in G\setminus\{I\}$, $\|g-I\|\geqslant p^{-\frac{2}{p-1}}$.

3 Algebraic convergence of non-elementary discrete subgroups

Lemma 6 Let

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad h=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\in\mathrm{PSL}(2,\mathbb{Q}_p)$$

with $gh = hg^{-1}$. Then g commutate with h.

Proof Since

$$gh = \begin{pmatrix} a\lambda & b/\lambda \\ c\lambda & d/\lambda \end{pmatrix}, \quad hg^{-1} = \begin{pmatrix} d\lambda & -b\lambda \\ -c/\lambda & a/\lambda \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Q}_p),$$

we see that either $\lambda^2 = -1$, namely $h = -z$, or if $\lambda^2 \neq \pm 1$, we have $b = c = 0$, $a = d$ namely $g(z) = I$. Hence g commutate with h.

Proof [Proof of the Theorem 1.] Since each G_n is discrete, we see that for any $I \neq g \in G_n$, $\|g - I\| \geqslant p^{-\frac{1}{p-1}}$. If the sequence $g_n \in G_n$ converges to $g \in G$, by ultrametric properties, we know that $\|g - I\| \geqslant p^{-\frac{2}{p-1}}$. Hence by Theorem 4, we see that G is discrete.

We claim that G is non-elementary. Suppose that G is elementary. By Theorem 1, either G is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{C}_p)$, every element of which leaves $\{0,\infty\}$ invariant and is therefore of the form $z \longrightarrow az^s$, $a \neq 0, s^2 = 1$, or G contains elliptic elements only.

If G contains elliptic elements only, then G is finite. Since each G_n is of $r-$generators and infinite, there exist two sequence $\{h_{n,1}\}$ and $\{h_{n,2}\}$ converging to the same element h, namely $h_{n,1} \circ h_{n,2}^{-1}$ tending to I. By ultrametric properties, we see that $h_{n,1} = h_{n,2}$, since G_n is discrete. This implies that G is infinite, namely G contains a loxodromic element at least.

If G contains elements leaving two points invariant, letting $\mathbb{H}$ be the union of all the finite extensions of $\mathbb{Q}_p$ of degree 2. Then each element in G is of the form $z \longrightarrow a^2 z^s$, $0 \neq a \in \mathbb{H}$, $s^2 = 1$.

First case, let

$$g_{n,1} = \begin{pmatrix} a_{n,1} & b_{n,1} \\ c_{n,1} & d_{n,1} \end{pmatrix}, g_{n,2} = \begin{pmatrix} a_{n,2} & b_{n,2} \\ c_{n,2} & d_{n,2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Q}_p)$$

tend to $A^2 z$ and $B^2 z$. Since

$$[g_{n,1}, g_{n,2}] = g_{n,1} g_{n,2} g_{n,1}^{-1} g_{n,2}^{-1} = \begin{pmatrix} a_{n,1}a_{n,2}d_{n,1}d_{n,2} + o(1) & o(1) \\ o(1) & a_{n,1}a_{n,2}d_{n,1}d_{n,2} + o(1) \end{pmatrix},$$

we see that $[g_{n,1}, g_{n,2}]$ tends to I. Since G_n is discrete, we know that $g_{n,1}$ commutate with $g_{n,2}$.

Second case, let

$$g_{n,1} = \begin{pmatrix} a_{n,1} & b_{n,1} \\ c_{n,1} & d_{n,1} \end{pmatrix}, g_{n,2} = \begin{pmatrix} a_{n,2} & b_{n,2} \\ c_{n,2} & d_{n,2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Q}_p)$$

tend to A^2/z and B^2/z. Since

$$[g_{n,1}, g_{n,2}] = g_{n,1} g_{n,2} g_{n,1}^{-1} g_{n,2}^{-1} = \begin{pmatrix} b_{n,1}c_{n,1}b_{n,2}c_{n,2} + o(1) & o(1) \\ o(1) & b_{n,1}c_{n,1}b_{n,2}c_{n,2} + o(1) \end{pmatrix},$$

we see that $[g_{n,1}, g_{n,2}]$ tends to I. Since G_n is discrete, we know that $g_{n,1}$ commutate with $g_{n,2}$.

The last case, let

$$g_{n,1} = \begin{pmatrix} a_{n,1} & b_{n,1} \\ c_{n,1} & d_{n,1} \end{pmatrix}, g_{n,2} = \begin{pmatrix} a_{n,2} & b_{n,2} \\ c_{n,2} & d_{n,2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Q}_p)$$

tend to $A^2 z$ and B^2/z. Since

$$[g_{n,1}, g_{n,2}] = g_{n,1} g_{n,2} g_{n,1}^{-1} g_{n,2}^{-1} = \begin{pmatrix} -a_{n,1}^2 b_{n,2}c_{n,2} + o(1) & o(1) \\ o(1) & -d_{n,2}^2 b_{n,2}c_{n,2} + o(1) \end{pmatrix},$$

we see that $[g_{n,1}, g_{n,2}]$ tends to $A^4 I$. Since G_n is discrete, we know that $[g_{n,1}, g_{n,2}]g_{n,1}^{-2} = I$, namely $g_{n,1}g_{n,2} = g_{n,2}g_{n,1}^{-1}$. By Lemma 6, we see that $g_{n,1}$ commutate with $g_{n,2}$.

In the end, since $g_{n,1}$ and $g_{n,2}$ are chosen arbitrarily, we see that each generator commutate with each other which implies that G_n is an elementary group. This is a contradiction.

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