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Lie symmetry analysis and conservation laws for the time fractional fourth-order evolution equation

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Abstract: In this paper, we study Lie symmetry analysis and conservation laws for the time fractional nonlinear fourth-order evolution equation. Using the method of Lie point symmetry, we provide the associated vector fields, and derive the similarity reductions of the equation, respectively. The method can be applied wisely and efficiently to get the reduced fractional ordinary differential equations based on the similarity reductions. Finally, by using the nonlinear self-adjointness method and Riemann-Liouville time-fractional derivative operator as well as Euler-Lagrange operator, the conservation laws of the equation are obtained.

Key words: Lie symmetry method; symmetry analysis; conservation laws

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四阶时间分数阶演化方程的 Lie 对称分析和守恒律

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摘 要：主要研究了四阶时间分数阶演化方程的 Lie 对称分析和守恒．基于 Lie 点对称方法，分别得到了该方程的相关向量场以及相似约化．在相似约化的基础上，通过该方法来获得分数阶常微分方程是非常有效的．最后，通过非线性的自伴随方法和时间分数阶的黎曼 – 刘维尔导数算子以及欧拉 – 拉格朗日算子，得到了该方程的守恒律.

关键词：Lie 对称方法；对称分析；守恒律

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1 Introduction

As we all know, the Lie group of transformations theory, was initially constructed by the Norwegian mathematician Sophus Lie in the early 19th century, which has been widely studied in differential equations[1−12], particularly in partial differential equations (PDEs) with multi-component systems. In recent years, it is important to investigate the fractional differential equations (FDEs) because of more and more applications in various fields[13−16]: systems identification, fluid flow, control problems, signal processing, viscoelastic materials, polymers, fluid mechanics, physics, economics, biology, engineering and so on. Actually, the symmetry analysis of FDEs and the fractional derivatives are proposed by Gazizov and his collaborators in [17]. Readers can refer to [18–22] for further details. In [23], the famous Noether theorem connects the symmetry theory with conservation laws of differential equations, which exist Euler-Lagrange equations. In addition, in order to look for conservation laws of FDEs, Frederico et al.[24−28] have provided fractional generalizations of Noether theorem. However, it's limited for many FDEs to deal with fractional generalizations. Then, a general theorem on conservation laws for arbitrary differential equations[29] was proposed by Ibragimov. Based on it, Lukashchuk[30] proposed Noether operators without fractional Lagrangians and proved fractional generalizations of the Noether operator. In [31], in order to get the conservation laws, the authors applied the fractional generalizations of the Noether operators to the time-fractional Kompaneets equations.

In this paper, we study the Lie symmetry analysis to the time fractional nonlinear fourth-order evolution equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+h_1u_xu_{xx}+h_2u^2u_{xx}+h_3uu_x^2+h_4u_{xxxx}=0,\ \ 0<\alpha\leqslant 2, \tag{1}$$

where $u=u(x,t)$, h_1, h_2, h_3 and h_4 are constants. When $\alpha=1$, $h_1=-2$, $h_2=-1$, $h_3=-2$, $h_4=1$, the time fractional equation becomes the following nonlinear fourth-order evolution equation

$$u_t-2u_xu_{xx}-u^2u_{xx}-2uu_x^2+u_{xxxx}=0. \tag{2}$$

Some new exact solutions of Eq. (2) are given in [32].

The paper is divided into five parts. In Section 2, some properties analyzing FDEs of Lie group method, and the associated infinitesimal generator are introduced. In Section 3, we obtain the corresponding similarity reduction of Eq. (1). Section 4 is devoted to construct the conservation laws. Finally , some conclusions and discussions are present.

2 Lie symmetry analysis

In this section, we investigate the time fractional nonlinear fourth-order evolution equation by means of Lie symmetry. First we recall, some basic properties of Lie symmetry analysis of FDEs. Here is the definition of Riemann-Liouville fractional partial derivative in [21].

Definition 1 [21] The Riemann-Liouville fractional partial derivative is defined by

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\begin{cases}\dfrac{\partial^m u}{\partial t^m}, & (\alpha=m\in\mathbb{N}),\\ \dfrac{1}{\Gamma(m-a)}\dfrac{\partial^m}{\partial t^m}\displaystyle\int_0^t\frac{u(\tau,x)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau, & (m-1<\alpha<m,\ \ m\in\mathbb{N}),\end{cases} \tag{3}$$

where ∂_t^m is the usual partial derivative of integer order *m*.

Consider a time fractional FDE with two independent variables

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F(x,t,u,u_x,u_{xx},\cdots), \alpha > 0. \tag{4}$$

One-parameter point transformations of Lie group is given by

$$\begin{aligned} x^* =& x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* =& t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u^* =& u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\ \frac{\partial^\alpha u^*}{\partial t^{*\alpha}} =& \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta^{\alpha t}(x,t,u) + O(\epsilon^2), \\ \frac{\partial^j u^*}{\partial x^{*j}} =& \frac{\partial^j u}{\partial x^j} + \epsilon\eta^{jx}(x,t,u) + O(\epsilon^2), j = 1,2,\cdots, \end{aligned} \tag{5}$$

where ϵ is the group parameter, and $\xi,\ \tau,\ \eta$ are infinitesimals. In addition, we consider the following infinitesimal generator

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}, \tag{6}$$

here,

$$\tau(x,t,u) = \frac{\mathrm{d}t^*}{\mathrm{d}\epsilon}\Big|_{\epsilon=0},\ \ \xi(x,t,u) = \frac{\mathrm{d}x^*}{\mathrm{d}\epsilon}\Big|_{\epsilon=0},\ \ \eta(x,t,u) = \frac{\mathrm{d}u^*}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}. \tag{7}$$

Based on the infinitesimal invariance criterion, we find the infinitesimal generator must satisfy

$$pr^{(\alpha,4)}V(\Delta_1)|_{\Delta_1=0} = 0, \tag{8}$$

where $\Delta_1 = u_t^\alpha + h_1 u_x u_{xx} + h_2 u^2 u_{xx} + h_3 u u_x^2 + h_4 u_{xxxx}$.

The prolongation operator $pr^{(\alpha,4)}$ is

$$pr^{\alpha,4}V = V + \xi_\alpha^0 \partial_{\partial_t^\alpha u} + \eta^x \partial_{u_x} + \eta^{xx}\partial_{u_{xx}} + \eta^{xxx}\partial_{u_{xxx}} + \eta^{xxxx}\partial_{u_{xxxx}}, \tag{9}$$

here,

$$\begin{aligned} \eta^x =& \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\ \eta^{xx} =& \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 \\ & - \tau_{uu}u_x^2 u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_t - 2\tau_u u_{xt}u_x, \\ \eta^{xxx} =& \eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x - \tau_{xxx}u_t + 3(\eta_{xuu} - \xi_{xxu})u_x^2 - 3\tau_{xxu}u_x u_t \\ & + (\eta_{uuu} - 3\xi_{xuu})u_x^3 + 3(\eta_{xu} - \xi_{xx})u_{xx} - 3\tau_{xx}u_{xt} - 3\tau_{xuu}u_x^2 u_t + 3(\eta_{uu} - 3\xi_{xu})u_x u_{xx} \\ & - 3\tau_{xu}u_{xx}u_t - 6\tau_{xu}u_{xt}u_x - 3\tau_x u_{xxt} + (\eta_u - 3\xi_x)u_{xxx} - \xi_{uuu}u_x^4 - 6\xi_{uu}u_{xx}u_x^2 - 3\tau_{uu}u_x^2 u_{xt} \\ & - \tau_{uuu}u_x^3 u_t - 3\xi_u u_{xx}^2 - 3\tau_u u_{xxt}u_x - 3\tau_u u_{xt}u_{xx} - 3\tau_{uu}u_t u_x u_{xx} - 4\xi_u u_x u_{xxx} - \tau_u u_t u_{xxx}, \\ \eta^{xxxx} =& \eta_{xxxx} + u_x(4\eta_{xxxu} - \xi_{xxxx}) - \tau_{xxxx}u_t + u_x^2(6\eta_{xxuu} - 4\xi_{xxxu}) + u_{xx}(6\eta_{xxu} - 4\xi_{xxx}) - 4\tau_{xxxu}u_t u_x \\ & - 4\tau_{xxx}u_{xt} + u_x^3(4\eta_{xuuu} - 6\xi_{xxuu}) + u_x u_{xx}(12\eta_{xuu} - 18\xi_{xxu}) - 6\tau_{xxuu}u_x^2 u_t - 6\tau_{xxu}u_{xx}u_t \\ & - 12\tau_{xxu}u_x u_{xt} + u_x^4(\eta_{uuuu} - 4\xi_{xuuu}) + u_x^2 u_{xx}(6\eta_{uuu} - 24\xi_{xuu}) + u_{xxx}(4\eta_{xu} - 6\xi_{xx}) - 6\tau_{xx}u_{xxt} \end{aligned}$$

$$
\begin{aligned}
&-4\tau_{xuuu}u_x^3u_t-12\tau_{xuu}u_tu_xu_{xx}-12\tau_{xuu}u_x^2u_{xt}+(3\eta_{uu}-12\xi_{xu})u_{xx}^2+u_xu_{xxx}(4\eta_{uu}-16\xi_{xu})\\
&-12\tau_{xu}u_{xx}u_{xt}-4\tau_{xu}u_tu_{xxx}-12\tau_{xu}u_xu_{xxt}-4\tau_xu_{xxxt}+u_{xxxx}(\eta_u-4\xi_x)-\xi_{uuuu}u_x^5\\
&-10\xi_{uuu}u_x^3u_{xx}-15\xi_{uu}u_xu_{xx}^2-10\xi_{uu}u_x^2u_{xxx}-4\tau_{uuu}u_x^3u_{xt}-12\tau_{uu}u_xu_{xx}u_{xt}-6\tau_{uu}u_x^2u_{xxt}\\
&-\tau_{uuuu}u_x^4u_t-6\tau_{uuu}u_x^2u_{xx}u_t-10\xi_uu_{xx}u_{xxx}-6\tau_uu_{xx}u_{xxt}-4\tau_uu_xu_{xxxt}\\
&-4\tau_uu_{xt}u_{xxx}-3\tau_{uu}u_tu_{xx}^2-4\tau_{uu}u_tu_xu_{xxx}-5\xi_uu_xu_{xxxx}-\tau_uu_tu_{xxxx}.
\end{aligned}\tag{10}
$$

From [18–21], we have

$$
\begin{aligned}
\xi_\alpha^0=&\frac{\partial^\alpha\eta}{\partial t^\alpha}+(\eta_u-\alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}\\
&+\mu+\sum_{n=1}^{\infty}\left[\binom{a}{n}\frac{\partial^\alpha\eta_u}{\partial t^\alpha}-\binom{a}{n+1}D_t^{n+1}(\tau)\right]\\
&\times D_t^{\alpha-n}(u)-\sum_{n=1}^{\infty}\binom{a}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x),
\end{aligned}\tag{11}
$$

where

$$
\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{a}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k},\tag{12}
$$

with the additional constraint condition

$$
\tau(x,t,u)|_{t=0}=0.\tag{13}
$$

In addition, when the infinitesimal η is linear to u, μ is invalid [19].

Now, we apply the above Lie symmetry analysis method above all to the Eq. (1), and obtain the following theorem.

Theorem 1 The vector fields of the time fractional fourth-order evolution equation is given by

$$
V_1=x\frac{\partial}{\partial x}+\frac{4t}{\alpha}\frac{\partial}{\partial t}-u\frac{\partial}{\partial u},\quad V_2=\frac{\partial}{\partial x}.\tag{14}
$$

Proof Substituting (9) and (10) into (1) , we find the following symmetry equation

$$
\xi_\alpha^0+h_4\eta^{xxxx}+h_1\eta^{xx}u_x+h_2u^2\eta^{xx}+h_1\eta^xu_{xx}+2h_3u\eta^xu_x+2h_2u\eta u_{xx}+h_3\eta u_x^2=0,\tag{15}
$$

by the condition that variables $u_x,\ u_t,\ u_{xt},\ u_{xx},\ u_{tt},\ u_{xxt},\ \cdots$ and $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$ for $n=1,2,\cdots$ of u are independent. Taking (10)-(13) into consideration, and making each power of the derivatives of u is equal to 0, we get the determining equations

$$
\begin{aligned}
&\xi_u=\xi_t=\tau_u=\tau_x=\eta_{uu}=0,\\
&h_4(\alpha\tau_t-4\xi_x)=0,\\
&\frac{\partial^\alpha\eta}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+h_2\eta_{xx}u^2=0,\\
&h_4(12\eta_{xuu}-18\xi_{xxu})+h_1(\eta_u+\alpha\tau_t-3\xi_x)-3h_2\xi_uu^2=0,\\
&\binom{a}{n}\frac{\partial^\alpha\eta_u}{\partial t^\alpha}-\binom{a}{n+1}D_t^{n+1}(\tau)=0,\ for\ \ n=1,2,\cdots
\end{aligned}\tag{16}
$$

Solving (16), we find

$$\begin{aligned} \xi &= a_1 x + a_2, \\ \tau &= \frac{4a_1}{\alpha} t, \\ \eta &= -a_1 u, \end{aligned} \tag{17}$$

where, a_1, a_2 are arbitrary constants. So the associated vector fields are given by

$$V_1 = x\frac{\partial}{\partial x} + \frac{4t}{\alpha}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial x}. \tag{18}$$

3 The similarity reduction

In this section, the similarity reduction for the time fractional nonlinear fourth-order evolution equation is obtained. Similar to the definition of group invariant solution of PDEs[4, 33], the group invariant solution of FDEs is defined.

Definition 2 [4, 33] If $u = v(x,t)$ satisfies Eq. (1), and $u = v(x,t)$ is an invariant surface of (6), i.e., it corresponds to the invariant surface condition $\tau(x,t,v)v_t + \xi(x,t,v)v_x = \eta(x,t,v)$, then $u = v(x,t)$ is an invariant solution of Eq. (1) relating to the infinitesimal generator of (6).

For the symmetry of V_1 of (18), we give the characteristic equation

$$\frac{\mathrm{d}x}{x} = \frac{\alpha \mathrm{d}t}{4t} = \frac{\mathrm{d}u}{-u}, \tag{19}$$

and the corresponding invariants are

$$z = xt^{-\frac{\alpha}{4}}, \quad u = t^{\frac{-\alpha}{4}} f(z). \tag{20}$$

Theorem 2 The time fractional nonlinear fourth-order evolution equation is reduced to the nonlinear ordinary differential equation(ODE) of time fractional order in the following by means of similarity transformation $u = t^{\frac{-\alpha}{4}} f(z)$ and the similarity variable $z = xt^{-\frac{\alpha}{4}}$,

$$\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha} f\right)(z) + h_1 f_z f_{zz} + h_2 f^2 f_{zz} + h_3 f (f_z)^2 + h_4 f_{zzzz} = 0, \tag{21}$$

with the Erdélyi-Kober fractional differential operator $P_\delta^{\tau,\alpha}$ of order[34]

$$\left(P_\delta^{\tau,\alpha} g\right)(z) := \prod_{j=0}^{m-1} \left(\tau + j - \frac{1}{\delta} z \frac{\mathrm{d}}{\mathrm{d}z}\right) \left(K_\delta^{\tau+\alpha,\ m-\alpha} g\right)(z), \quad z > 0,\ \delta > 0,\ \alpha > 0, \tag{22}$$

$$m = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases} \tag{23}$$

where

$$\left(K_\delta^{\tau,\alpha} g\right)(z) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^\infty (v-1)^{\alpha-1} v^{-(\tau+\alpha)} g(zv^{\frac{1}{\delta}}) \mathrm{d}v, & \alpha > 0, \\ g(z), & \alpha = 0, \end{cases} \tag{24}$$

is the Erdélyi-Kober fractional integral operator.

Proof Let $n-1<\alpha<n,\ \ n=1,2,\cdots$, then we obtain the similarity transformation of Riemann-Liouville fractional derivatives as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{-\frac{\alpha}{4}}f(xs^{-\frac{\alpha}{4}})\mathrm{d}s\right]. \tag{25}$$

If $v=\frac{t}{s}$, we get

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}=&\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1}v^{-(n+1-\frac{5\alpha}{4})}f(zv^{\frac{\alpha}{4}})\mathrm{d}v\right]\\ =&\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z)\right].\end{aligned} \tag{26}$$

Considering the relation of $z=xt^{-\frac{\alpha}{4}}$, we get

$$t\frac{\partial}{\partial t}\Phi(z)=tx\Big(-\frac{\alpha}{4}\Big)t^{-\frac{\alpha}{4}-1}\Phi'(z)=-\frac{\alpha}{4}z\frac{\mathrm{d}}{\mathrm{d}z}\Phi(z). \tag{27}$$

Hence, the simplification of the above equation is

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z)\right]=&\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z)\right)\right]\\ =&\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{5\alpha}{4}}\ \left(n-\frac{5\alpha}{4}-\frac{\alpha}{4}z\frac{\mathrm{d}}{\mathrm{d}z}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z)\right)\right].\end{aligned} \tag{28}$$

Continuously, we have

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z)\right]=&\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z)\right)\right]\\ =&\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{5\alpha}{4}}\ \left(n-\frac{5\alpha}{4}-\frac{\alpha}{4}z\frac{\mathrm{d}}{\mathrm{d}z}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z)\right)\right]\\ =&\cdots=t^{\frac{-5\alpha}{4}}\prod_{j=0}^{n-1}\left(1-\frac{5\alpha}{4}+j-\frac{\alpha}{4}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z).\end{aligned} \tag{29}$$

Considering the fractional differential operator of Erdélyi-Kober fractional operator of (22), we get

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{5\alpha}{4}}\left(K_{\frac{4}{\alpha}}^{1-\frac{\alpha}{4},n-\alpha}f\right)(z)\right]=t^{-\frac{5\alpha}{4}}\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}f\right)(z). \tag{30}$$

Thus,

$$\frac{\partial^\alpha u}{\partial t^\alpha}=t^{-\frac{5\alpha}{4}}\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}f\right)(z). \tag{31}$$

Therefore, it's fascinating that Eq. (1) reduces to the fractional-order ODE

$$\left(P_{\frac{4}{\alpha}}^{1-\frac{5\alpha}{4},\alpha}f\right)(z)+h_1f_zf_{zz}+h_2f^2f_{zz}+h_3f(f_z)^2+h_4f_{zzzz}=0. \tag{32}$$

4 Conservation laws

4.1 Preliminaries of conservation laws

According to the Riemann-Liouville left-sided time-fractional derivative[35−37], we have

$$ {}_0D_t^\alpha u = D_t^n({}_0I_t^{n-\alpha}u), \tag{33}$$

in Eq. (1), where D_t is the operator concerning t of differentiation, $n=[\alpha]+1$. In addition, the definition of ${}_0I_t^{n-\alpha}u$ is given as follows.

Definition 3 [22] The left-sided time-fractional integral of order ${}_0I_t^{n-\alpha}u$ is defined by

$$({}_0I_t^{n-\alpha}u)(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(\theta,x)}{(t-\theta)^{1-n+\alpha}}\mathrm{d}\theta, \tag{34}$$

where, $\Gamma(z)$ is the Gamma function.

Definition 4 A conserved vector is said to obey the conservation law, if it satisfies the following conservation equation

$$D_t(C^t)+D_x(C^x)=0, \tag{35}$$

where $C^t=C^t(t,x,u,\cdots),\ \ C^x=C^x(t,x,u,\cdots)$.

Firstly, we suppose that Eq. (1) has the following formal Lagrangian

$$L = v(x,t)\left[u_t^\alpha + h_1u_xu_{xx} + h_2u^2u_{xx} + h_3uu_x^2 + h_4u_{xxxx}\right], \tag{36}$$

where, $v(x,t)$ is a new dependent variable. In view of the previous Lagrangian, we get an action integral as follows

$$\int_0^T\int_\Omega L\left(x,\ t,\ u,\ v,\ D_t^\alpha(u),\ u_x,\ \cdots\right)\mathrm{d}x\mathrm{d}t. \tag{37}$$

Similar to the case of integer-order nonlinear differential equations[30−31], the adjoint equation is available. Therefore, we get the conclusion that the Euler-Lagrange equation of nonlinear fourth-order evolution equation Eq. (1) with adjoint equation in the following

$$\frac{\delta L}{\delta u}=0, \tag{38}$$

and the definition of the Euler-Lagrange operator is obtained as below.

Definition 5 [30−31] The Euler-Lagrange operator is defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^*\frac{\partial}{\partial D_t^\alpha u} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{3x}} + D_x^4\frac{\partial}{\partial u_{xxxx}}, \tag{39}$$

where $(D_t^\alpha)^*$ is the adjoint operator of (D_t^α).

For the Riemann-Liouville differential operators

$$(D_t^\alpha)^* = (-1)^n I_T^{n-\alpha}(D_t^n) \equiv_t^C D_T^\alpha, \tag{40}$$

here, $I_T^{n-\alpha}$ is the right-sided operator of fractional integration of order $n-\alpha$, and $I_T^{n-\alpha}$ is defined by

$$I_T^{n-\alpha}f(t,x) = \frac{1}{\Gamma(n-\alpha)}\int_t^T \frac{f(\tau,x)}{(\tau-t)^{1+\alpha-n}}\mathrm{d}\tau,\ \ n=[\alpha]+1. \tag{41}$$

We find that Eq. (1.1) with the Riemann-Liouville fractional derivative satisfies the conservation equation of Eq. (33) as follows

$$C^t = D_t^{n-1}({}_0I_t^{n-\alpha}u),\;\; C^x = h_4 u_{xxx} + h_2 u^2 u_x + \frac{h_1}{2}u_x^2 \tag{42}$$

when, $h_3 = 2h_2$.

In case of two independent variables t, x, and one dependent variable $u(x,t)$, we get

$$\overline{X} + D_t(\tau)l + D_x(\xi)l = W\frac{\delta}{\delta u} + D_t N^t + D_x N^x, \tag{43}$$

where l is the identity operator, $\frac{\delta}{\delta u}$ shows the Euler-Lagrange operator, and the Noether operators are provided by N^t, N^x respectively. In addition, $\overline{X}$ is defined by

$$\overline{X} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \xi_\alpha^0\frac{\partial}{\partial D_t^\alpha u} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{xxxx}\frac{\partial}{\partial u_{xxxx}}, \tag{44}$$

and

$$W = \eta - \tau u_t - \xi u_x. \tag{45}$$

Applying the Riemann-Liouville time-fractional derivative to Eq. (1), we obtain the following operator N^t defined in [30–31]

$$N^t = \tau l + \sum_{k=0}^{n-1}(-1)^k\,{}_0D_t^{\alpha-1-k}(W)D_t^k\frac{\partial}{\partial({}_0D_t^\alpha u)} - (-1)^n J\left(W, D_t^n\frac{\partial}{\partial({}_0D_t^\alpha u)}\right), \tag{46}$$

where J is the integral,

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-n}}d\mu\mathrm{d}t. \tag{47}$$

The operator N^x is defined by

$$\begin{aligned} N^x =&\xi l + W\left(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{xx}} + D_x^2\frac{\partial}{\partial u_{xxx}} - D_x^3\frac{\partial}{\partial u_{xxxx}}\right)\\ &+ D_x(W)\left(\frac{\partial}{\partial u_{xx}} - D_x\frac{\partial}{\partial u_{xxx}} + D_x^2\frac{\partial}{\partial u_{xxxx}}\right)\\ &+ D_x^2(W)\left(\frac{\partial}{\partial u_{xxx}} - D_x\frac{\partial}{\partial u_{xxxx}}\right)\\ &+ D_x^3\frac{\partial}{\partial u_{xxxx}}. \end{aligned} \tag{48}$$

No matter what the solutions and generator X of Eq. (1) are, we obtain

$$\left(\overline{X}L + D_t(\tau)L + D_x(\xi)L\right)\big|_{(1)} = 0. \tag{49}$$

It is easy to find that Euler-Lagrange (38) of Eq. (1) is equal to zero. Hence, the conservation law resulted ftom the right-hand side of the equality is given by [30]

$$D_t(N^tL) + D_x(N^xL) = 0. \tag{50}$$

Comparing (35) and (50), we find that the following components of conserved vectors is always valid obviously

$$C^t = N^t L, \quad C^x = N^x L. \tag{51}$$

4.2 Conservation laws of time fractional foam drainage equation

In this section,the conservation laws of Eq. (1) is available.

Theorem 3 The components of conserved vectors about fourth-order evolution equation is given by the following two cases

Case 1 when $\alpha \in (0, 1)$

$$\begin{aligned}
V_1 : C_1^t &= {}_0D_t^{\alpha-1}(W_1)v + J(W_1, v_t)\\
&= {}_0D_t^{\alpha-1}\Big(-u - \frac{4t}{\alpha}u_t - xu_x\Big)v + J\Big(-u - \frac{4t}{\alpha}u_t - xu_x, v_t\Big),\\
C_1^x &= -\Big(u + \frac{4t}{\alpha}u_t + xu_x\Big)[2v(h_3 - h_2)uu_x - v_x(h_1u_x + h_2u^2)]\\
&\quad - v\Big(2u_x + \frac{4t}{\alpha}u_{xt} + xu_{xx}\Big)(h_1u_x + h_2u^2)\\
&\quad + h_4v_{xxx}\Big(u + \frac{4t}{\alpha}u_t + xu_x\Big) - h_4v_{xx}\Big(2u_x + \frac{4t}{\alpha}u_{xt} + xu_{xx}\Big)\\
&\quad + h_4v_x\Big(3u_{xx} + \frac{4t}{\alpha}u_{xxt} + xu_{xxx}\Big) - h_4v\Big(4u_{xxx} + \frac{4t}{\alpha}u_{xxxt} + xu_{xxxx}\Big).
\end{aligned} \tag{52}$$

$$\begin{aligned}
V_2 : C_2^t &= {}_0D_t^{\alpha-1}(W_2)v + J(W_2, v_t)\\
&= {}_0D_t^{\alpha-1}(-u_x)v + J(-u_x, v_t),\\
C_2^x &= -u_x[2v(h_3 - h_2)uu_x - v_x(h_1u_x + h_2u^2)] - vu_{xx}(h_1u_x + h_2u^2)\\
&\quad + h_4u_xv_{xxx} - h_4u_{xx}v_{xx} + h_4u_{xxx}v_x - h_4u_{xxxx}v.
\end{aligned} \tag{53}$$

Case 2 when $\alpha \in (1, 2)$

$$\begin{aligned}
V_1 : C_1^t &= {}_0D_t^{\alpha-1}(W_1)v + J(W_1, v_t) - {}_0D_t^{\alpha-2}(W_1)v_t - J(W_1, v_{tt})\\
&= {}_0D_t^{\alpha-1}\Big(-u - \frac{4t}{\alpha}u_t - xu_x\Big)v + J\Big(-u - \frac{4t}{\alpha}u_t - xu_x, v_t\Big)\\
&\quad - {}_0D_t^{\alpha-2}\Big(-u - \frac{4t}{\alpha}u_t - xu_x\Big)v_t - J\Big(-u - \frac{4t}{\alpha}u_t - xu_x, v_{tt}\Big),\\
C_1^x &= -\Big(u + \frac{4t}{\alpha}u_t + xu_x\Big)[2v(h_3 - h_2)uu_x - v_x(h_1u_x + h_2u^2)]\\
&\quad - v\Big(2u_x + \frac{4t}{\alpha}u_{xt} + xu_{xx}\Big)(h_1u_x + h_2u^2)\\
&\quad + h_4v_{xxx}\Big(u + \frac{4t}{\alpha}u_t + xu_x\Big) - h_4v_{xx}\Big(2u_x + \frac{4t}{\alpha}u_{xt} + xu_{xx}\Big)\\
&\quad + h_4v_x\Big(3u_{xx} + \frac{4t}{\alpha}u_{xxt} + xu_{xxx}\Big) - h_4v\Big(4u_{xxx} + \frac{4t}{\alpha}u_{xxxt} + xu_{xxxx}\Big).
\end{aligned} \tag{54}$$

$$\begin{aligned}
V_2 : C_2^t &= {}_0D_t^{\alpha-1}(W_2)v + J(W_2, v_t) - {}_0D_t^{\alpha-2}(W_2)v_t - J(W_2, v_{tt})\\
&= {}_0D_t^{\alpha-1}(-u_x)v + J(-u_x, v_t) - {}_0D_t^{\alpha-2}(-u_x)v_t - J(-u_x, v_{tt}),\\
C_2^x &= -u_x\left[2v(h_3 - h_2)uu_x - v_x(h_1u_x + h_2u^2)\right] - vu_{xx}(h_1u_x + h_2u^2)\\
&\quad + h_4u_xv_{xxx} - h_4u_{xx}v_{xx} + h_4u_{xxx}v_x - h_4u_{xxxx}v,
\end{aligned} \tag{55}$$

where the functions of $W_i,\, i=1,2$ have the form

$$W_1=-u-\frac{4t}{\alpha}u_t-xu_x,\quad W_2=-u_x. \tag{56}$$

Proof Substituting (46) and (48) into Eq. (1), we get the conserved vector as below.

For the case of $\alpha\in(0,1)$,

$$\begin{aligned}C_i^t=&\tau L+(-1)^0\,{}_0D_t^{\alpha-1}(W_i)D_t^0\frac{\partial L}{\partial\left({}_0D_t^\alpha u\right)}-(-1)J\left(W_i,D_t\frac{\partial L}{\partial\left({}_0D_t^\alpha u\right)}\right)\\=&\,{}_0D_t^{\alpha-1}(W_i)v+J(W_i,v_t),\end{aligned} \tag{57}$$

and

$$\begin{aligned}N^x=&\xi l+W\left(\frac{\partial}{\partial u_x}-D_x\frac{\partial}{\partial u_{xx}}+D_x^2\frac{\partial}{\partial u_{xxx}}-D_x^3\frac{\partial}{\partial u_{xxxx}}\right)\\&+D_x(W)\left(\frac{\partial}{\partial u_{xx}}-D_x\frac{\partial}{\partial u_{xxx}}+D_x^2\frac{\partial}{\partial u_{xxxx}}\right)\\&+D_x^2(W)\left(\frac{\partial}{\partial u_{xxx}}-D_x\frac{\partial}{\partial u_{xxxx}}\right)\\&+D_x^3\frac{\partial}{\partial u_{xxxx}}.\end{aligned} \tag{58}$$

For the case of $\alpha\in(1,2)$,

$$\begin{aligned}C_i^t=&\tau L+(-1)^0\,{}_0D_t^{\alpha-1}(W_i)D_t^0\frac{\partial L}{\partial\left({}_0D_t^\alpha u\right)}-(-1)^1J\left(W_i,D_t^1\frac{\partial L}{\partial\left({}_0D_t^\alpha u\right)}\right)\\&+(-1)^1\,{}_0D_t^{\alpha-2}(W_i)D_t^1\frac{\partial L}{\partial\left({}_0D_t^\alpha u\right)}-(-1)^2J\left(W_i,D_t^2\frac{\partial L}{\partial\left({}_0D_t^\alpha u\right)}\right)\\=&\,{}_0D_t^{\alpha-1}(W_i)v+J(W_i,v_t)-{}_0D_t^{\alpha-2}(W_i)v_t-J(W_i,v_{tt}),\end{aligned} \tag{59}$$

and

$$\begin{aligned}N^x=&\xi l+W\left(\frac{\partial}{\partial u_x}-D_x\frac{\partial}{\partial u_{xx}}+D_x^2\frac{\partial}{\partial u_{xxx}}-D_x^3\frac{\partial}{\partial u_{xxxx}}\right)\\&+D_x(W)\left(\frac{\partial}{\partial u_{xx}}-D_x\frac{\partial}{\partial u_{xxx}}+D_x^2\frac{\partial}{\partial u_{xxxx}}\right)\\&+D_x^2(W)\left(\frac{\partial}{\partial u_{xxx}}-D_x\frac{\partial}{\partial u_{xxxx}}\right)\\&+D_x^3\frac{\partial}{\partial u_{xxxx}},\end{aligned} \tag{60}$$

here

$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+v_x\frac{\partial}{\partial v}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+v_{xx}\frac{\partial}{\partial v_x}+v_{xt}\frac{\partial}{\partial v_t}+\cdots \tag{61}$$

and $u=u(x,t),\ v=v(x,t)$.

For the operators $V_1=x\frac{\partial}{\partial x}+\frac{4t}{\alpha}\frac{\partial}{\partial t}-u\frac{\partial}{\partial u},\quad V_2=\frac{\partial}{\partial x}$, referring to (45), we get the Lie characteristics functions, $W_1=-u-\frac{4t}{\alpha}u_t-xu_x,\ W_2=-u_x$.

In accordance with the fractional generalizations of the Riemann-Liouville differential operators[30], we have the specific form of the time fractional fourth-order evolution equationa as follows. When $\alpha\in(0,1)$, we substitute (36) along with (56) into (57) and (58). When $\alpha\in(1,2)$, we substitute (36) and (56) into (59) and (60), and see that Theorem 3 is 1.00,0.00,0.00. The proof is completed.

5 Conclusions and discussions

In this paper, we study the time fractional nonlinear fourth-order evolution equation by way of Lie symmetry. Based on known results, we conclude the corresponding vector fields. In addition, these corresponding vector fields are further applied in order to construct the symmetry reductions of $\frac{\partial^\alpha u}{\partial t^\alpha} + h_1 u_x u_{xx} + h_2 u^2 u_{xx} + h_3 u u_x^2 + h_4 u_{xxxx}, 0 < \alpha \leqslant 2$. In the end, we obtain the conservation laws. However, we still have a lot of problems to research, such as how can we study the time-space fractional equation ? How to solve the fractional differential equations with more independent and dependent variables? These are our future work.

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