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The controllability of nonlinear fractional damped dynamical systems with control delay

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Abstract: This paper deals with the controllability of nonlinear fractional damped dynamical system with control delay which involves fractional Caputo derivatives. Using fixed point theorem, we establish sufficient conditions for the controllability of nonlinear fractional damped dynamical system with control delay are established. These conditions are easily checked. Two examples are given to illustrate the main results.

Key words: controllability; fractional damped systems; control delay

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具有控制时滞的非线性分数阶阻尼系统的可控性

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摘　要：讨论了在 Caputo 意义下，控制项具有时滞的非线性分数阶阻尼系统的可控性. 利用不动点定理，得到了非线性分数阶阻尼系统可控的充分条件，所得的条件易于验证. 并且给出两个例子说明所得结论的可行性.

关键词：可控性；分数阶阻尼系统；控制时滞

1 Introduction

The fractional differential equations have gained considerable attention in the past four decades, which play an important role in physics, chemistry and engineering, see, for instance, [1–5]. Controllability is a fundamental concept in control theory. Recently, the controllability of fractional differential equations have been extensively

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studied, and many results are obtained[6−8]. Notice that time delay is a common phenomenon in practical systems, many researchers have paid much attention and have also achieved many great accomplishments[10−11].

A remarkable feature for control delay systems is that the future evolution of the systems depends not only on the present control state, but also on a period of control history.

In [12], Sebakhy and Bayoumi explored a simplified criterion for the controllability of linear systems with delay in control. In [13], Chyung obtained a necessary and sufficient condition for the controllability of linear time-invariant systems with a time delay.

Balachandran, Zhou, Tuujillo and Kokila studied the relative controllability of several kinds of fractional dynamical systems with delays in [14–16].

The fractional damped system is investigated in [9], where the controllability of linear and nonlinear fractional damped dynamical systems are explored. In [17], He, Zhou and Kou investigated the linear fractional damped dynamical system with time delay in control

$$ {}_0^C D_t^\alpha x(t) - A {}_0^C D_t^\beta x(t) = Bu(t) + Cu(t-\tau), t \geqslant 0, $$

where $A {}_0^C D_t^\beta x(t)$ is the fractional damped term, u is the control input, and τ is the time control delay, and a necessary and sufficient condition for the controllability was established.

Motivated mainly by [9, 17], we concern with, in this paper, the controllability of the following nonlinear fractional damped dynamical system with delays in control:

$$ \begin{cases} {}_0^C D_t^\alpha x(t) - A {}_0^C D_t^\beta x(t) = Bu(t) + C(t-\tau) + f(t,x,u), t \geqslant 0, \\ x(0) = x_0, x'(0) = x_0', \\ u(t) = \psi(t), -\tau \leqslant t \leqslant 0, \end{cases} \tag{1} $$

where $0 < \beta \leqslant 1 < \alpha \leqslant 2$, $x \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a control vector, $A \in \mathbb{R}^{n\times n}$,$B, C \in \mathbb{R}^{n\times m}$ are any matrices, τ is the time delay, and $\psi(t)$ is the initial control function, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function. ${}_0^C D_t^\alpha x, {}_0^C D_t^\beta x$ denotes an order of α and β Caputo fractional derivative of x respectively.

The Caputo fractional derivative of order $\alpha(0 \leqslant m \leqslant \alpha < m+1)$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha+1)} \int_0^t \frac{f^{(m+1)}(\theta)}{(t-\theta)^{\alpha-m}} \mathrm{d}\theta. $$

The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by

$$ E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+1)}, \alpha > 0, z \in \mathbb{C}. $$

The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ $\alpha, \beta > 0$ is defined by

$$ E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+\beta)}, \alpha > 0, z \in \mathbb{C}. $$

Definition 1 The set $y(t) = \{x(t), x'(t), u_t\}$ is the complete state of the system (1) at time t.

Definition 2 The system (1) is said to be controllable on $J = [0, T]$, if, for each complete state $y(0)$ and for each vector $x_1 \in \mathbb{R}^n$, there exists a control $u \in C(J)$ such that the corresponding solution of (1) with $x(0) = x_0, x'(0) = x_0'$ satisfies $x(T) = x_1$.

In this definition of controllability, we are concerned in steering only the states but not the velocity vector x_0' in (1).

Lemma 1 [9] The general solution of system

$$\begin{cases} {}_0^C D_t^\alpha x(t) - A {}_0^C D_t^\beta x(t) = f(t), t \geqslant 0, \\ x(0) = x_0, x'(0) = x_0', \end{cases}$$

with $0 < \beta \leqslant 1 < \alpha \leqslant 2$ can be written as

$$\begin{aligned} x(t) = {} & E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' \\ & + \int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s)\mathrm{d}s, \end{aligned}$$

where $f : J \to \mathbb{R}^n$ is a continuous function.

For linear system

$$\begin{cases} {}_0^C D_t^\alpha x(t) - A {}_0^C D_t^\beta x(t) = Bu(t) + Cu(t-\tau), t \geqslant 0, \\ x(0) = x_0, x'(0) = x_0', \end{cases} \tag{2}$$

we have the following lemmas.

Lemma 2 [17] The linear fractional damped dynamical system with control delay (2) is controllable if and only if the controllability Grammian matrix defined by

$$\begin{aligned} W(t) = {} & \int_0^{t-\tau} [(((t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})B \\ & +(t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})C)(((t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})B \\ & +(t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})C)^\top)]\mathrm{d}s \\ & + \int_{t-\tau}^t ((t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta}))BB^\top((t-s)^{\alpha-1}(E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta}))^\top)\mathrm{d}s \end{aligned} \tag{3}$$

is invertible.

Lemma 3 [17] The linear fractional damped dynamical system with control delay (2) is controllable if and only if

$$\operatorname{rank}[B, AB, A^2B, \cdots, A^{n-1}B, C, AC, A^2C, \cdots, A^{n-1}C] = n. \tag{4}$$

2 Main Results

In this section, we state and prove our main results.

Denote

$$Q = \{(z, v) : z \in C_n(J),\ v \in C_m(J)\},$$

with the uniform norm $\|(z, v)\| = \|z\| + \|v\| = \max\limits_{t\in J}|z(t)| + \max\limits_{t\in J}|v(t)|$, where $C_n(J) = \{f : J \to \mathbb{R}^n | f$ is continuous on $J\}$, $C_m(J) = \{f : J \to \mathbb{R}^m | f$ is continuous on $J\}$ are Banach spaces. We denote the max norms in

$\mathbb{R}^n$ and $\mathbb{R}^m$ by $|\cdot|_n$ and $|\cdot|_m$, and use the notation $|\cdot|$ if there is no confusion. For each $(z,v)\in Q$, consider the following nonlinear fractional damped dynamical system:

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) - A_0^C D_t^{\beta} x(t) = Bu(t) + Cu(t-\tau) + f(t,z,v),\ t \geqslant 0, \\ x(0)=x_0, x'(0)=x_1, \\ u(t)=\psi(t), -\tau \leqslant t \leqslant 0. \end{cases} \tag{5}$$

By Lemma 1, the solution of the system (5) can be written as

$$\begin{aligned} x(t) = {} & E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' \\ & + \int_{-\tau}^{0}(t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})C\psi(s)\mathrm{d}s \\ & + \int_{0}^{t}(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s,z,v)\mathrm{d}s \\ & + \int_{0}^{t-\tau}[(((t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta}))B \\ & \qquad +(t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})C)]u(s)\mathrm{d}s. \\ & + \int_{t-\tau}^{t}(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})Bu(s)\mathrm{d}s. \end{aligned} \tag{6}$$

For brevity, let us introduce the following notations and constants:

$$\begin{aligned} & G_1(t,s) = (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})B + (t-s-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s-\tau)^{\alpha-\beta})C, \\ & G_2(t,s) = (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})B, \\ & a_1 = \sup_{t\in[0,T]}\|E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0'\|, \\ & a_2 = \sup_{t,s\in[0,T]}\|E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\|, \\ & a_3 = \sup_{t\in[0,T]}\Big\|\int_{-\tau}^{0}(t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})C\psi(s)\mathrm{d}s\Big\|, \\ & a_4 = \max_{i=1,2}\sup_{t\in[0,T]}\|G_i^{\top}(T,t)\|, \\ & a_5 = \sup_{t\in[0,T]}\Big\|\int_{0}^{t-\tau}[(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})B + (t-\tau-s)^{\alpha-1} \\ & \quad \times E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})C]\mathrm{d}s + \int_{t-\tau}^{t}(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})B\mathrm{d}s\Big\|, \\ & d_1 = 4a_4|W^{-1}|(|x_1|+a_1+a_3), \qquad d_2 = 4(a_1+a_3), \\ & c_1 = 4a_2a_4T^{\alpha}|W^{-1}|\alpha^{-1}, \qquad c_2 = 4a_2T^{\alpha}\alpha^{-1}, \\ & c = \max\{a_5c_1, c_1, c_2\}, \qquad d = \max\{a_5d_1, d_1, d_2\}, \\ & \sup|f| = \sup\{|f(s,z(s),v(s))|; s\in J\}. \end{aligned} \tag{7}$$

We also define

$$\begin{aligned} \eta(y(0),x_1;z,v) = {} & x_1 - E_{\alpha-\beta}(AT^{\alpha-\beta})x_0 + AT^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(AT^{\alpha-\beta})x_0 - TE_{\alpha-\beta,2}(AT^{\alpha-\beta})x_0' \\ & - \int_{-\tau}^{0}(T-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(T-s-\tau)^{\alpha-\beta})C\psi(s)\mathrm{d}s \end{aligned}$$

$$- \int_0^T (T-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta}) f(s,z,v)\mathrm{d}s,$$

and define the control function

$$u(t) = \begin{cases} G_1^\top(T,t)W^{-1}\eta(y(0),x_1;z,v), & t \in [0, T-\tau), \\ G_2^\top(T,t)W^{-1}\eta(y(0),x_1;z,v), & t \in [T-\tau, T], \end{cases} \tag{8}$$

where the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily.

In order to prove our main result, we need the following lemma.

Lemma 4 [18] Suppose the function f is bounded locally in ν and satisfies that

$$\lim_{|\nu|\to\infty} \frac{|f(w,\nu)|}{|\nu|} = 0$$

uniformly in $w \in J$. Then for every pair of constants c, d, there is a constant r such that if $|\nu| \leqslant r$, then $c|f(w,\nu)| + d \leqslant r$ for all $w \in J$.

Theorem 1 Suppose that the continuous function f satisfies the condition

$$\lim_{|(x,u)|\to\infty} \frac{|f(t,x,u)|}{|(x,u)|} = 0 \tag{9}$$

uniformly in $t \in J$, and the linear fractional system (2) is controllable. Then the nonlinear system (1) is controllable on J.

Proof By hypothesis, system (2) is controllable. It follows from Lemma 2 that W given by (3) is invertible. We define the operator $\Psi:\ Q \to Q$ as follow:

$$\Psi(z,v) = (x,u),\ \ (z,v) \in Q,$$

where $u(t)$ is given by (8), and

$$\begin{aligned} x(t) = {} & E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' \\ & + \int_{-\tau}^0 (t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})C\psi(s)\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s,z,v)\mathrm{d}s \\ & + \int_0^{t-\tau} [((t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})B \\ & + (t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})C)u(s)]\mathrm{d}s. \\ & + \int_{t-\tau}^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})Bu(s)\mathrm{d}s. \end{aligned} \tag{10}$$

We denote $u_i(t) = G_i^\top(T,t)W^{-1}\eta(y(0),x_1;z,v), i = 1,2$. Then

$$\begin{aligned} u_i(t) &= G_i^\top(T,t)W^{-1}\eta(y(0),x_1;z,v) \\ &= G_i^\top(T,t)W^{-1}(x_1 - E_{\alpha-\beta}(AT^{\alpha-\beta})x_0 + AT^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(AT^{\alpha-\beta})x_0 \\ &\quad - TE_{\alpha-\beta,2}(AT^{\alpha-\beta})x_0' - \int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(s,z,v)\mathrm{d}s \\ &\quad - \int_{-\tau}^0 (T-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(T-s-\tau)^{\alpha-\beta})C\psi(s)\mathrm{d}s). \end{aligned} \tag{11}$$

Now, we show that there exists a constant $r > 0$ such that

$$\Psi(Q(r)) \subset Q(r),$$

where $Q(r) = \Big\{(z,v) \in Q : \|z\| \leqslant \dfrac{r}{2} \text{ and } \|v\| \leqslant \dfrac{r}{2}\Big\}$.

By (11) and (10), we have

$$\begin{aligned} |u_i(t)| &\leqslant \|G_i^\top(T,t)\||W^{-1}|\big[|x_1| + a_1 + a_3 + a_2 T^\alpha \alpha^{-1} \sup|f|\big] \\ &\leqslant a_4|W^{-1}|[|x_1| + a_1 + a_3] + a_4|W^{-1}|a_2 T^\alpha \alpha^{-1} \sup|f| \\ &\leqslant \frac{d_1}{4} + \frac{c_1}{4}\sup|f|, \qquad i = 1,2, \end{aligned} \tag{12}$$

and

$$\begin{aligned} |x(t)| &\leqslant a_1 + a_3 + a_5|u_i(s)| + a_2\int_0^t (t-s)^{\alpha-1}\sup|f|\mathrm{d}s \\ &\leqslant \frac{d_2}{4} + a_5\Big[\frac{d_1}{4} + \frac{c_1}{4}\sup|f|\Big] + a_2 T^\alpha \alpha^{-1}\sup|f| \\ &\leqslant \frac{d}{2} + \frac{c}{2}\sup|f|. \end{aligned} \tag{13}$$

Since the function f satisfies (9), by Lemma 4, for each pair of positive constants c and d, there exists a positive constant r such that, if $|(\overline{z},\overline{v})| \leqslant r$, then

$$c|f(t,\overline{z},\overline{v})| + d \leqslant r, \text{for all } t \in [0,T]. \tag{14}$$

Now we take c, d as given by (7), and choose r such that (14) holds. Therefore, if $\|z\| \leqslant \dfrac{r}{2}$ and $\|v\| \leqslant \dfrac{r}{2}$, then $|z(s)| + |v(s)| \leqslant r$ for all $s \in [0,T]$. It follows that $d + c\sup|f| \leqslant r$. Therefore, by (12), we have $|u(s)| \leqslant \dfrac{r}{4}$ for all $s \in [0,T]$, and hence $\|u\| \leqslant \dfrac{r}{4}$, by (13), $\|x\| \leqslant \dfrac{r}{2}$. Thus, $\Psi(Q(r)) \subset Q(r)$.

Next, we show that Ψ has a fixed point in $Q(r)$. Actually, the continuity of f implies that the operator Ψ is continuous. By Arzela-Ascoli theorem, we have that the operator Ψ is completely continuous. Since $Q(r)$ is closed, bounded and convex, it follows from the Schauder fixed point theorem that Ψ admits a fixed point $(z,v) \in Q(r)$ such that $\Psi(z,v) = (z,v) \equiv (x,u)$. Hence $x(t)$ is the solution of the system (1), and it is easy to verify that $x(T) = x_1$, this means that the control function $u(t)$ steers the system (1) from complete state $y(0)$ to x_1 on $[0,T]$. Thus the system (1) is controllable on $[0,T]$.

For $\rho_i \in L^1(J),\ i = 1,2,\cdots,q$, $\|\rho_i\|$ is the L^1 norm of $\rho_i(\cdot)$ defined by $\|\rho_i\| = \int_J |\rho_i(s)|\mathrm{d}s$. We use the notations

$$e_i = \max\{4a_2a_4T^\alpha|W^{-1}|\alpha^{-1}\|\rho_i\|, 4a_2T^\alpha\alpha^{-1}\|\rho_i\|\}, \text{ and } \overline{c}_i = \max\{a_5e_i, e_i\},\ \ i = 1,2,\cdots,q.$$

Suppose the function f satisfies

$$|f(t,x,u)| \leqslant \sum_{i=1}^{q}\rho_i(t)\phi_i(x,u), \tag{15}$$

where $\phi_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ are measurable functions and $\rho_i : J \to \mathbb{R}_+$ are L^1 functions for $i = 1,2,\cdots,q$.

Theorem 2 Suppose that the linear system (2) is controllable and the following condition holds

$$\lim_{r\to\infty}\sup\Big(r-\sum_{i=1}^{q}\overline{c}_i\sup\phi_i(x,u):\|(x,u)\|\leqslant r\Big)=+\infty. \tag{16}$$

Then the nonlinear fractional damped system (1) is controllable on J.

Proof Define the operator $\Phi: Q\to Q$ by $\Phi(z,v)=(x,u)$ where x and u are as defined in (10) and (11). Now let

$$\psi_i(r)=\sup\{\phi_i(z,v):\|(z,v)\|\leqslant r\}.$$

By (16), there exists $r_0>0$ such that

$$r_0-\sum_{i=1}^{q}\overline{c}_i\psi_i(r_0)\geqslant d,$$

where d is defined in (7). This implies that

$$\sum_{i=1}^{q}\overline{c}_i\psi_i(r_0)+d\leqslant r_0.$$

Then, by (11) and (10), we have

$$\begin{aligned}|u_i(t)|&\leqslant\|G_i^{\top}(T,t)\||W^{-1}|\Big[|x_1|+a_1+a_3+a_2T^{\alpha}\alpha^{-1}\sum_{i=1}^{q}\|\rho_i\|\psi_i(r_0)\Big]\\&\leqslant a_4|W^{-1}|[|x_1|+a_1+a_3]+a_4|W^{-1}|a_2T^{\alpha}\alpha^{-1}\sum_{i=1}^{q}\|\rho_i\|\psi_i(r_0)\\&\leqslant\frac{d_1}{4}+\frac{e_i}{4}\sum_{i=1}^{q}\psi_i(r_0)\\&\leqslant\frac{d}{4}+\frac{\overline{c}_i}{4}\sum_{i=1}^{q}\psi_i(r_0),\end{aligned}$$

and

$$\begin{aligned}|x(t)|&\leqslant a_1+a_3+a_5|u(t)|+a_2\int_0^t(t-s)^{\alpha-1}\sum_{i=1}^{q}\|\rho_i\|\psi_i(r_0)\mathrm{d}s\\&\leqslant\frac{d_2}{4}+a_5\Big[\frac{d_1}{4}+\frac{e_i}{4}\sum_{i=1}^{q}\psi_i(r_0)\Big]+a_2T^{\alpha}\alpha^{-1}\sum_{i=1}^{q}\|\rho_i\|\psi_i(r_0)\\&\leqslant\frac{d}{2}+\frac{\overline{c}_i}{2}\sum_{i=1}^{q}\psi_i(r_0).\end{aligned}$$

Therefore, $|u(t)|\leqslant\dfrac{r_0}{4}$ for all $t\in J$, hence, $\|u\|\leqslant\dfrac{r_0}{4}$, which gives $\|x\|\leqslant\dfrac{r_0}{2}$. Thus, $\Phi(Q(r_0))\subset Q(r_0)$. The next step is the same as that in Theorem 1, so it is omitted.

3 Examples

In this section we apply the results obtained in the previous sections for the following fractional damped dynamical systems with delays in control.

Example 1 Consider the fractional damped nonlinear system with control delay

$${}_0^C D_t^{1.5}x(t)-\begin{pmatrix}3&0\\0&4\end{pmatrix}{}_0^C D_t^{0.5}x(t)=\begin{pmatrix}2\\0\end{pmatrix}u(t)+\begin{pmatrix}0\\1\end{pmatrix}u(t-1)+f(t,x,u) \tag{17}$$

where $f(t,x,u)$ is given by

$$f(t,x,u)=\left(\frac{1+x_1}{1+x_1^2+u_1^2},\frac{1+x_2}{1+x_2^2+u_2^2}\right)^{\top}.$$

where $x,u\in\mathbb{R}^2$. A simple computation shows that

$$\begin{pmatrix}B & AB & C & AC\end{pmatrix}=\begin{pmatrix}2 & 6 & 0 & 0\\ 0 & 0 & 1 & 4\end{pmatrix}$$

and

$$\operatorname{rank}\begin{pmatrix}B & AB & C & AC\end{pmatrix}=2.$$

From Lemma 3, we know that the corresponding linear system of (17) is controllable.

We see that $f(t,x,u)$ is continuous and satisfies the condition (9), and thus the fractional damped system (17) is controllable.

Example 2 Consider the nonlinear system

$$ {}_0^CD_t^{\alpha}x(t)-A{}_0^CD_t^{\beta}x(t)=Bu(t)+Cu(t-\tau)+f(t,x,u), \tag{18}$$

where $\alpha=1.5,\beta=0.5$,

$$A=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix},B=\begin{pmatrix}1\\ 0\end{pmatrix},C=\begin{pmatrix}0\\ 3\end{pmatrix} \tag{19}$$

and $f(t,x,u)$ is given by

$$f(t,x,u)=\left(\frac{x_1}{1+x_1+u_1},\frac{x_2}{1+x_2+u_2}\right)^{\top}. \tag{20}$$

where $x,u\in\mathbb{R}^2$. A simple computation shows that

$$\begin{pmatrix}B & AB & C & AC\end{pmatrix}=\begin{pmatrix}1 & 2 & 0 & 0\\ 0 & 0 & 3 & 9\end{pmatrix} \tag{21}$$

and

$$\operatorname{rank}\begin{pmatrix}B & AB & C & AC\end{pmatrix}=2. \tag{22}$$

From Lemma 3, we know that the linear system is controllable. Furthermore,

$$|f(t,x,u)|\leqslant\frac{1}{\|x(t)\|},\quad (t,x,u)\in J\times\mathbb{R}^n\times\mathbb{R}^m.$$

In order to prove the required results, it is enough to show the condition(16) holds under the following settings:

$$q=1,\quad \phi(x,u)=\frac{1}{x}.$$

Hence,

$$\lim_{r\to\infty}\sup\left(r-\sup c\frac{1}{x}\right)=+\infty,$$

and thus the fractional damped system (18) is controllable on $[0,T]$.

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