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Approximate controllability of semilinear fractional evolution equations of order $\alpha \in (1, 2]$ with finite delay

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Abstract: This paper considers the approximate controllability of semilinear fractional evolution equations of order $\alpha \in (1, 2]$ with finite delay. Using the contraction mapping principle, we explore the existence and uniqueness of the mild solution. Furthermore, under certain hypotheses, the approximate controllability is obtained by the theory of strongly continuous α-order cosine family. As an illustration of the application of the obtained result, an example is given at last.

Key words: approximate controllability; fractional evolution equation; finite delay; α-order cosine family; contraction mapping principle

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$\alpha \in (1, 2]$ 阶的有限时滞半线性发展方程的近似可控性

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摘 要：讨论了 $\alpha \in (1, 2]$ 阶的有限时滞半线性发展方程的近似可控性. 首先运用压缩映像原理证明了弱解的存在唯一性，进而在适当条件下运用 α-阶强连续余弦族理论证明了系统的近似可控性.

关键词：近似可控性；分数阶发展方程；有限时滞； α-阶余弦族；压缩映像原理

1 Introduction

In the past few decades, fractional differential systems have played an important role in physics, chemistry and engineering, etc.. Regarded as the generalization of integer order systems, fractional order systems are more suitable

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for many experimental results that the integer order differential systems can't describe[1−3]. There exists numerous anomalous diffusion phenomena in real life. In order to model these phenomena accurately, fractional evolution equations are established by introducing the fractional differential operator in the standard diffusion systems. Many researchers studied the fundamental issues of fractional evolution equations and many valuable results were obtained. For example, El-Borai[4] investigated the existence of the solutions of fractional evolution equations by using the Laplace transform. Balachandran and Park[5] considered the existence of the mild solutions of a class of nonlocal Cauchy problem for abstract fractional semilinear evolution equations by using fractional calculus theory and the fixed point theorem. Zhou and Jiao[6−7] discussed the existence of the mild solutions to fractional evolution and neutral evolution equations in a Banach space in which the mild solutions are introduced based on probability density function and semigroup. The existence of mild solutions of fractional evolution equations is the foundation for the research of other properties of the related systems.

Moreover, we note that the study of controllability is essential for many engineering and physics problems. There are a lot of research works on the exact controllability and approximate controllability. But, the concept of exact controllability is too strong and the approximate controllability is more appropriate for practical application. There are many results on the controllability of fractional evolution equations of order $\alpha \in (0, 1]$, see [8–12]. In these papers, the approximate controllability of semilinear fractional evolution equations of order $\alpha \in (0, 1]$, with or without delay, were studied. But, to the best of our knowledge, the contributions on the controllability of semilinear fractional evolution equations of order $\alpha \in (1, 2]$ are few. In [13–15], Li et al. studied the controllability of diverse fractional systems of order $\alpha \in (1, 2]$ by using fixed point theorem and operator theory. And Shukla et al.[16] investigated the approximate controllability of semilinear fractional control system of order $\alpha \in (1, 2]$ with infinite delay by sequence method.

Motivated by the above discussion, in this paper we study the approximate controllability for a class of fractional systems of order $\alpha \in (1, 2]$ with finite delay in Banach space. Under a weak Lipschitz condition, some sufficient conditions are obtained to guarantee the approximate controllability of the systems. Consider the following semilinear fractional evolution equation

$$\begin{cases} {}_0^C D_t^\alpha y(t) = Ay(t) + Bv(t) + f(t, y_t), \ \ t \in (0, T], \\ y_0 = \phi, \ \ y_0' = \psi, \end{cases} \tag{1}$$

where $\alpha \in (1, 2]$, $0 < T < \infty$, ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative, $A : D(A) \subset X \to X$ is a closed and densely defined operator in Banach space X and A is an infinitesimal generator of a strongly continuous α-order cosine family $\{C_\alpha(t)\}_{t \geqslant 0}$. Here, the state $y(\cdot)$ takes values in Banach space X and the control function $v(\cdot)$ is given in $L_2([0, T]; U)$, U is a Banach space of admissible control functions. B is a bounded linear operator from U to X. In addition, $f : [0, T] \times C([-h, 0]; X) \to X$ is nonlinear, which will be specified later; $\phi(\cdot), \psi(\cdot) \in C([-h, 0]; X)$. For continuous function $y : [-h, T] \to X$, we define $y_t : [-h, 0] \to X$ as $y_t(\theta) = y(t + \theta)$, $\theta \in [-h, 0]$, $t \in [0, T]$. The corresponding linear system of the system (1) is

$$\begin{cases} {}_0^C D_t^\alpha x(t) = Ax(t) + Bu(t), \ \ t \in (0, T], \\ x(0) = \phi(0), \ \ x'(0) = \psi(0). \end{cases} \tag{2}$$

This paper is organized as follows. In section 2, we recall some necessary concepts and give the definition of the mild solution of the system (1). Then the existence of the mild solution and the approximate controllability of the semilinear fractional control systems are discussed in section 3. In section 4, we give an example to illustrate the effectiveness of our result.

2 Preliminaries

Let X be a Banach space with norm $\|\cdot\|_X$ and $\mathfrak{L}(X)$ be the space of all bounded linear operators on X. Let $L_p([0,T];X)$, $1 \leqslant p < \infty$, be the space of X-valued Bochner integrable function $f:[0,T] \to X$ with the norm

$$\|f\|_{L_p} = \Big(\int_0^T \|f(t)\|_X^p \mathrm{d}t\Big)^{\frac{1}{p}}.$$

$C([-h,0];X)$ and $C([-h,T];X)$ are X-valued continuous functional space with the norms

$$\|y_t\|_{C([-h,0];X)} = \sup_{-h\leqslant\theta\leqslant 0} \|y_t(\theta)\|_X \ \text{ for } \ t \in [0,T].$$

$$\|y\|_{C([-h,T];X)} = \sup_{-h\leqslant t\leqslant T} \|y(t)\|_X.$$

$C^n([0,T];X)$ is the space of X-valued functions continuously differentiable at least to the n^{th} order.

In the following, we recall some definitions to be used throughout this paper.

Definition 1 [2] The Riemann-Liouville fractional integral ${}_0I_t^\alpha x$ of order $\alpha > 0$ for a function $x(\cdot) \in L_1([0,T];X)$ is defined as

$${}_0I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\mathrm{d}s, \ \ t \in [0,T],$$

where Γ is the gamma function.

Definition 2 [2] The Caputo fractional derivative ${}_0^CD_t^\alpha x$ of order $\alpha > 0$ for a function $x(\cdot) \in L_1([0,T];X) \bigcap C^n([0,T];X)$ is defined as

$${}_0^CD_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)\mathrm{d}s, \ \ t \in [0,T],$$

where $n = [\alpha]+1$. In particular, if $\alpha \in (1,2)$, one has

$${}_0^CD_t^\alpha x(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}x''(s)\mathrm{d}s, \ \ t \in [0,T].$$

Suppose that $Bu = 0$, $\psi(0) = 0$, $\phi(0) = \eta \in D(A)$ in the system (2), then it is equivalent to

$$\begin{cases} {}_0^CD_t^\alpha x(t) = Ax(t), \\ x(0) = \eta, \ \ x'(0) = 0. \end{cases} \tag{3}$$

Following [17], we have the following definition.

Definition 3 [17] A family $\{C_\alpha(t)\}_{t\geqslant 0} \subset \mathfrak{L}(X)$ is called the solution operator (or strongly continuous α-order fractional cosine family) for the system (3), if the following conditions are satisfied:

(a)$C_\alpha(t)$ is strongly continuous for $t \geqslant 0$ and $C_\alpha(0) = I$;

(b)$C_\alpha D(A) \subset D(A)$ and $AC_\alpha(t)\eta = C_\alpha(t)A\eta$ for all $\eta \in D(A)$, $t \geqslant 0$;

(c)$C_\alpha(t)\eta$ is a solution of (3) for all $\eta \in D(A)$.

A is called the infinitesimal generator of $C_\alpha(t)$.

Definition 4 [17] The α-order cosine family $C_\alpha(t)$ is called exponentially bounded if there are constants $M \geqslant 1$ and $\omega \geqslant 0$ such that

$$\|C_\alpha(t)\|_X \leqslant M\mathrm{e}^{\omega t}, \ \ t \geqslant 0. \tag{4}$$

If the solution operator $C_\alpha(t)$ of (3) satisfies (4), we shall say $A \in \mathcal{C}^\alpha(X;M,\omega)$.

We define the corresponding fractional sine family $S_\alpha : [0,\infty) \to \mathfrak{L}(X)$ by

$$S_\alpha(t) = \int_0^t C_\alpha(s)\mathrm{d}s, \ \ t \geqslant 0,$$

and let the associated fractional Riemann-Liouville family be $P_\alpha : [0,\infty) \to \mathfrak{L}(X)$ such that

$$P_\alpha(t) =_0 I_t^{\alpha-1} C_\alpha(t) \ \ t \geqslant 0.$$

Taking the Laplace transform to (1), by the property of Laplace transform, we have

Definition 5 [13, 17] A function $y(\cdot) \in C([-h,T];X)$ is said to be a mild solution of the system (1) if $y_0 = \phi, \ y_0' = \psi$ and

$$y(t) = C_\alpha(t)\phi(0) + S_\alpha(t)\psi(0) + \int_0^t P_\alpha(t-s)[Bv(s) + f(s,y_s)]\mathrm{d}s, \ \ t \in (0,T]. \tag{5}$$

Let $y(T;\phi(0),\psi(0),v)$ be the state value of the system (1) at time T corresponding to the control v initial value $\phi(0)$. The system (1) is said to be approximately controllable in $[0,T]$, if for every desired final state y_1 and constant $\epsilon > 0$, there exists a control function $v \in L_2([0,T],U)$ such that

$$\|y(T;\phi(0),\psi(0),v) - y_1\|_X < \epsilon.$$

Moreover, let $K_T(f) = \{y(T;\phi(0),\psi(0),v); v \in U\}$. We see that if $\overline{K_T(f)} = X$, then the system (1) is approximately controllable, where $\overline{K_T(f)}$ denote the closure of $K_T(f)$. In particular, if $\overline{K_T(0)} = X$, then the system (2) is approximately controllable.

3 Main Results

In this section, the existence of the mild solution and the approximate controllability for the system (1) are proved under the following hypotheses:

- (H_1) There exists a function $l(\cdot) \in L_{\frac{1}{\beta}}([0,T],[0,\infty))$, such that

  $$\|f(t,x_t) - f(t,y_t)\|_X \leqslant l(t)\|x_t - y_t\|_{C([-h,0];X)},$$

  where $\beta \in (0,1]$, $x_t(\cdot), y_t(\cdot) \in C([-h,0],X)$, $t \in [0,T]$ and $l(t) \leqslant N$, for every $t \in [0,T]$, $N \in \mathbb{R}^+$;
- (H_2) The operator A belongs to $\mathcal{C}^\alpha(X;M,0)$, i.e., there exists a constant $M \geqslant 1$ such that $\|C_\alpha(t)\|_X \leqslant M, \ \ t \in [0,T]$;
- (H_3) The operator B is invertible, and $B^{-1} : R(f) \subseteq X \to U$;
- (H_4) As $\lambda \to 0^+$, $\lambda R(\lambda, \Lambda^{\mathrm{T}}) \to 0$ in the strong operator topology, where

  $$R(\lambda,\Lambda^{\mathrm{T}}) = \left(\lambda I + \Lambda^{\mathrm{T}}\right)^{-1}, \ \lambda > 0, \ \ \Lambda^{\mathrm{T}} = \int_0^T P_\alpha(T-s)BB^*P_\alpha^*(T-s)\mathrm{d}s,$$

  and B^*, $P_\alpha^*(t)$ are the adjoint operators of B and $P_\alpha(t)$, respectively.

Similar to the arguments in [8, 18], we see that the hypothesis (H_4) holds if and only if the corresponding linear system (2) is approximately controllable on $[0,T]$.

First, we prove the existence and uniqueness of mild solution of the system (1).

Theorem 1 Suppose that (H_1) and (H_2) hold. Then (5) admits a unique mild solution of the system (1) in $C([-h,T];X)$ with $y_0=\phi,\ y_0'=\psi$ for each control function $v(\cdot)\in L_2([0,T];U)$, provided that

$$\frac{MNT^\alpha}{\Gamma(\alpha)}<1.$$

Proof Define the mapping $\Phi: C([-h,T];X)\to C([-h,T];X)$ as

$$\begin{cases}(\Phi y)(t)=C_\alpha(t)\phi(0)+S_\alpha(t)\psi(0)+\int_0^t P_\alpha(t-s)[Bv(s)+f(s,y_s)]\mathrm{d}s,\ \ t\in(0,T],\\ (\Phi y_0)=\phi.\end{cases}$$

Let $l=\max\limits_{0\leqslant t\leqslant T}\|f(t,0)\|_X$, $\|B\|_X\leqslant M_B$, $r=\Big(1-\frac{MNT^\alpha}{\Gamma(\alpha)}\Big)^{-1}\Big(M\|\phi\|_{C([-h,0];X)}+MT\|\psi\|_{C([-h,0];X)}+\frac{MlT^\alpha}{\Gamma(\alpha)}+\frac{MM_BT^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\|v\|_{L_2}\Big)$, and $B_r=\{y(\cdot)\in C([-h,T];X):\|y\|_{C([-h,T];X)}\leqslant r, y_0=\phi\}$, which is a bounded and closed subset of $C([-h,T];X)$. For any $y\in B_r$, we have

$$\|y_t\|_{C([-h,0];X)}\leqslant\|y\|_{C([-h,T];X)}\leqslant r.$$

Now, we prove Φ has a fixed point in the space $C([-h,T];X)$. We first show that Φ maps B_r into itself. Obviously, $\Phi y_0\in B_r$. For $t\in(0,T]$, we have

$$\begin{aligned}\|(\Phi y)(t)\|_X&\leqslant\|C_\alpha(t)\phi(0)\|_X+\|S_\alpha(t)\psi(0)\|_X+\int_0^t\|P_\alpha(t-s)[Bv(s)+f(s,y_s)]\|_X\mathrm{d}s\\&\leqslant M\|\phi\|_{C([-h,0];X)}+MT\|\psi\|_{C([-h,0];X)}+\frac{MT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\|f(s,y_s)-f(s,0)\|_X\mathrm{d}s\\&\quad+\frac{MT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\|f(s,0)\|_X\mathrm{d}s+\frac{MT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\|Bv(s)\|_X\mathrm{d}s\\&\leqslant M\|\phi\|_{C([-h,0];X)}+MT\|\psi\|_{C([-h,0];X)}+\frac{MNT^\alpha}{\Gamma(\alpha)}\|y_s\|_{C([-h,0];X)}\\&\quad+\frac{MlT^\alpha}{\Gamma(\alpha)}+\frac{MM_BT^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\|v\|_{L_2}\\&\leqslant M\|\phi\|_{C([-h,0];X)}+MT\|\psi\|_{C([-h,0];X)}+\frac{MNT^\alpha r}{\Gamma(\alpha)}+\frac{MlT^\alpha}{\Gamma(\alpha)}+\frac{MM_BT^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\|v\|_{L_2}\\&=r,\end{aligned}$$

which means $\|\Phi y\|_{C([-h,T];X)}\leqslant r$, $\Phi y\subseteq B_r$.

Next, we show that Φ is a contraction mapping on B_r. In fact, for any $y(\cdot),z(\cdot)\in B_r$,

$$\begin{aligned}\|(\Phi y)(t)-(\Phi z)(t)\|_X&\leqslant\int_0^t\|P_\alpha(t-s)\|\|f(s,y_s)-f(s,z_s)\|_X\mathrm{d}s\\&\leqslant\frac{MT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t l(t)\|y_s-z_s\|_{C([-h,0];X)}\mathrm{d}s\\&\leqslant\frac{MNT^\alpha}{\Gamma(\alpha)}\|y_t-z_t\|_{C([-h,0];X)}\\&\leqslant\frac{MNT^\alpha}{\Gamma(\alpha)}\|y-z\|_{C([-h,T];X)}.\end{aligned}$$

That is

$$\|\Phi y-\Phi z\|_{C([-h,T];X)}\leqslant\frac{MNT^\alpha}{\Gamma(\alpha)}\|y-z\|_{C([-h,T];X)}.$$

Since $\dfrac{MNT^{\alpha}}{\Gamma(\alpha)} < 1$, Φ is a contraction mapping. Therefore, Φ has the unique fixed point in the space $C([-h,T];X)$. (5) is the unique mild solution of the system (1) and the proof is completed.

Now, we prove the approximate controllability of the system (1) when $B = I$, where I is the identity operator on X. In this case, $U \equiv X$. The semilinear system (1) can be rewritten as

$$\begin{cases} {}_0^C D_t^{\alpha} y(t) = Ay(t) + v(t) + f(t, y_t), \ \ t \in (0,T], \\ y_0 = \phi, \ \ y_0' = \psi, \end{cases} \tag{6}$$

and the corresponding linear system is

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = Ax(t) + u(t), \ \ t \in (0,T], \\ x(0) = \phi(0), \ x'(0) = \psi(0). \end{cases} \tag{7}$$

Theorem 2 Suppose that (H_1), (H_2) and (H_4) with $B \equiv I$ hold. Then the semilinear system (6) is approximately controllable on $[0,T]$.

Proof From the argument above, we know that if the hypothesis (H_4) with $B \equiv I$ is satisfied, then the corresponding linear system (7) is approximately controllable on $[0,T]$, i.e., $\overline{K_T^{(7)}(0)} = X$.

Consider the following semilinear system:

$$\begin{cases} {}_0^C D_t^{\alpha} y(t) = Ay(t) + f(t, y_t) + u(t) - f(t, x_t), \ \ t \in (0,T], \\ y_0 = \phi, \ \ y_0' = \psi. \end{cases} \tag{8}$$

By comparing (6) and (8), setting $v(t) \equiv u(t) - f(t, x_t)$ for $t \in [0,T]$, we get that the approximate controllability of the system (6) on $[0,T]$ is equivalent to that of the system (8). Thus we only need to show that the system (8) is approximately controllable on $[0,T]$.

For this purpose, here we shall show that $K_T^{(8)}(f) \supset K_T^{(7)}(0)$.

By (5), we obtain that the mild solution of (8) is

$$\begin{cases} y(t) = C_{\alpha}(t)\phi(0) + S_{\alpha}(t)\psi(0) + \displaystyle\int_0^t P_{\alpha}(t-s)[u(s) - f(s, x_s) + f(s, y_s)]\mathrm{d}s, \ \ t \in (0,T], \\ y_0 = \phi. \end{cases}$$

Let $x(\cdot) \in C([0,T];X)$ be the mild solution of the system (7) excited by some control inputs u. Take $x(\cdot) \in C([-h,T];X)$ to be

$$\begin{cases} x(t) = C_{\alpha}(t)\phi(0) + S_{\alpha}(t)\psi(0) + \displaystyle\int_0^t P_{\alpha}(t-s)u(s)\mathrm{d}s, \ \ t \in (0,T], \\ x_0 = \phi. \end{cases}$$

Then we obtain that

$$x_0 - y_0 = 0, \tag{9}$$

and

$$x(t) - y(t) = \int_0^t P_{\alpha}(t-s)[f(s, x_s) - f(s, y_s)]\mathrm{d}s, \ \ t \in (0,T].$$

By (H_1) and (H_2), we have

$$\begin{aligned}\|x(t)-y(t)\|_X &\leqslant \int_0^t \|P_\alpha(t-s)[f(s,x_s)-f(s,y_s)]\|_X \mathrm{d}s \\ &\leqslant \frac{MT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t l(t)\|x_s-y_s\|_{C([-h,0];X)}\mathrm{d}s \\ &\leqslant \frac{MNT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|x_s-y_s\|_{C([-h,0];X)}\mathrm{d}s \\ &\leqslant \frac{MNT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|x-y\|_{C([-h,T];X)}\mathrm{d}s.\end{aligned} \tag{10}$$

Then it follows from (9) and (10) that

$$\|x-y\|_{C([-h,T];X)} \leqslant \frac{MNT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|x-y\|_{C([-h,T];X)}\mathrm{d}s.$$

Hence Gronwall's inequality leads to $x(t)=y(t)$ for all $t\in[-h,T]$. That is $K_T^{(8)}(f)\supset K_T^{(7)}(0)$, which means that the system (6) is approximately controllable on $[0,T]$ and the proof is completed.

Finally, we study the approximate controllability of the system (1) when $B\neq I$.

Theorem 3 Suppose that (H_1)–(H_4) hold. Then the semilinear system (1) is approximately controllable on $[0,T]$.

Proof Similar to the argument in Theorem 2, we consider the following semilinear system:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = Ay(t)+f(t,y_t)+Bu(t)-f(t,x_t), \ \ t\in(0,T], \\ y_0=\phi, \ \ y_0'=\psi. \end{cases} \tag{11}$$

Under the hypothesis (H_3), we can find a control function v to satisfy $Bv(t)\equiv Bu(t)-f(t,x_t)$. Then the approximate controllability of the semilinear system (1) is equivalent to $K_T^{(11)}(f)\supset K_T^{(2)}(0)$.

Note that the mild solution of (11) is

$$\begin{cases} y(t)=C_\alpha(t)\phi(0)+S_\alpha(t)\psi(0)+\displaystyle\int_0^t P_\alpha(t-s)[Bu(s)-f(s,x_s)+f(s,y_s)]\mathrm{d}s, \ \ t\in(0,T], \\ y_0=\phi. \end{cases}$$

Let $x(\cdot)\in C([0,T];X)$ be the mild solution of the system (2) excited by some control inputs u. Take $x(\cdot)\in C([-h,T];X)$ to be

$$\begin{cases} x(t)=C_\alpha(t)\phi(0)+S_\alpha(t)\psi(0)+\displaystyle\int_0^t P_\alpha(t-s)Bu(s)\mathrm{d}s, \ \ t\in(0,T], \\ x_0=\phi. \end{cases}$$

Then we see that

$$x_0-y_0=0,$$

and

$$x(t)-y(t)=\int_0^t P_\alpha(t-s)[f(s,x_s)-f(s,y_s)]\mathrm{d}s, \ \ t\in(0,T].$$

Similar to the proof of the theorem 2, based on the hypotheses (H_1) and (H_2), we have

$$\|x-y\|_{C([-h,T];X)} \leqslant \frac{MNT^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|x-y\|_{C([-h,T];X)}\mathrm{d}s.$$

It then follows from Gronwall's inequality that $x(t)=y(t)$ for all $t\in[-h,T]$, and we get that $K_T^{(11)}(f)\supset K_T^{(2)}(0)$. this implies that the system (1) is approximately controllable on $[0,T]$ and the proof is completed.

4 Example

In this section, we give an example to illustrate the effectiveness of our result.

Consider the following partial differential equation

$$\begin{cases} {}_0^C D_t^\alpha y(t,z) = y_{zz}(t,z) + f(t, y(t+\theta, z)) + Bu(t,z), \ \ t \in (0,T], \ \ \theta \in [-h,0], \ \ z \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0, \ \ t \in (0,T], \\ y(\theta,z) = \phi(\theta,z), \ \ \theta \in [-h,0], \ \ z \in [0,\pi], \\ y_t(\theta,z) = \psi(\theta,z), \ \ \theta \in [-h,0], \ \ z \in [0,\pi], \end{cases} \tag{12}$$

where $\alpha \in (1,2]$, $\phi(\theta,z), \psi(\theta,z)$ are continuous functions.

Let $X = \mathfrak{L}[0,\pi]$, define $A : D(A) \subset X \to X$ by

$$Ae = e'',$$

where $D(A) = \{e \in X : e, e'$ are absolutely continuous, $e'' \in X, \ \ e(0) = e(\pi) = 0\}$. Put $e_n(z) = \sqrt{\left(\frac{2}{\pi}\right)}\sin(nz)$; $z \in [0,\pi]$, $n = 1, 2, \cdots$, then $\{e_n, n = 1, 2, \cdots\}$ is an orthogonal set X and e_n is the eigenfunction corresponding to eigenvalue $-n^2$ of operator A. Then operator A can be given by

$$Ae = \sum_{n=1}^{\infty} -n^2 (e, e_n) e_n, \ \ e \in D(A).$$

It is obvious that A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t \geqslant 0}$, defined on X, which is given by

$$C(t)e = \sum_{n=1}^{\infty} \cos(nt)(e, e_n) e_n, \ \ e \in X,$$

and the associated sine family is given by

$$S(t)e = \sum_{n=1}^{\infty} \frac{1}{n} \sin(nt)(e, e_n) e_n, \ \ e \in X.$$

For $\alpha \in (1,2)$, A is the infinitesimal generator of a strongly continuous exponentially bounded fractional cosine family $\{C_\alpha(t)\}_{t \geqslant 0}$ such that $C_\alpha(0) = I$(see Theorem 3.1, [17]), and

$$C_\alpha(t) = \int_0^\infty \varphi_{t,\frac{\alpha}{2}}(s) C(s) \mathrm{d}s, \ \ t > 0,$$

where $\varphi_{t,\frac{\alpha}{2}}(s) = t^{-\frac{\alpha}{2}} \Phi_{\frac{\alpha}{2}}(st^{-\frac{\alpha}{2}})$, and

$$\Phi_\gamma(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n! \Gamma(-\gamma n + 1 - \gamma)}, \ \ 0 < \gamma < 1.$$

Then the fractional sine family $\{S_\alpha\}_{t \geqslant 0}$ and the fractional Riemann-Liouville family $\{P_\alpha\}_{t \geqslant 0}$ can be given correspondingly. Therefore, the hypothesis (H_4) holds. For every $t \in [0,T]$, let $f(t, y(t+\theta, z)) = f(t, y_t)$, we can take $f(t, y_t) = t^{-\frac{1}{3}} \|y_t\|$, then the hypothesis (H_1) is satisfied. Furthermore, we assume that $M = \sup_{t \in [0,\infty)} \|C_\alpha(t)\| + 1$, $B = I$ and $D(B) = R(f)$, then the hypotheses (H_2) and (H_3) are satisfied. Hence, the system (12) is approximately controllable.

5 Conclusions

By extending the [10], we discussed the approximate controllability for a class of semilinear fractional evolution equations of order $\alpha \in (1, 2]$ with finite delay. We considered the existence and uniqueness of the mild solution of the system and obtained some sufficient conditions of the approximate controllability of the system.

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