DOI:10.3969/J.ISSN.100-5137.2017.03.007

Two-interval even order differential operators in direct sum spaces with inner product multiples

Suo Jianqing[1*], Wang Wanyi[2]

(1.School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China;

2.College of Science, Inner Mongolia Agricultural University, Hohhot 010019, China)

Abstract: We study two-interval singular differential equations and show that their self-adjoint operator realizations in direct sum Hilbert spaces can be enlarged by using inner product multiples.

Key words: self-adjoint; interface conditions; transmission conditions

CLC number: O 175.3 **Document code:** A **Article ID:** 1000-5137(2017)03-0393-13

2010 MSC: 34B24, 34L99, 47E05

直和空间中带有内积倍数的两区间偶数阶微分算子

索建青 [1*], 王万义 [2]

(1. 内蒙古大学 数学科学学院，呼和浩特 010021；2. 内蒙古农业大学 理学院，呼和浩特 010019)

摘　要：研究两区间奇异微分方程的自伴算子实现，证明了在直和空间中通过运用内积倍数可以扩大自伴算子实现的范围.

关键词：自伴；交接条件；转移条件

1 Introduction

Partly motivated by applications, in particular (see [1–2]), Everitt and Zettl in [3] developed a theory of self-adjoint realizations of Sturm-Liouville problems on two intervals in the direct sum of Hilbert spaces associated with these intervals. See Chapter 13 in [4] for an exposition of this theory. This theory was extended in [5] to higher order regular and singular equations and any number of intervals, finite or infinite.

As in the one interval case the characterization of [5] depends on maximal domain vectors. These vectors depend on the coefficients of each differential equation and this dependence is implicit and complicated. In [6]

Received date: 2016-11-21

Foundation item: This work is supported by Natura Science Foundation of China (11361039); Specialized Research Fund for the Doctoral Program of Higher Education (20131501120006).

∗**Corresponding author:** Suo Jianqing, lecturer, research area: spectral theory of differential operators, E-mail: sjq.hello@163.com

Wang, Sun and Zettl give an explicit characterization of all self-adjoint domains in terms of certain solutions for real λ for the one interval case when one endpoint is regular and the other is singular. In analogy with the celebrated Weyl limit-point (LP), limit-circle (LC) theory in the second order case, they construct LC and LP solutions and characterize the self-adjoint domains in terms of the LC solutions. Following [6], Hao, Wang, Sun and Zettl give a new characterization by dividing (a_1, b_1) into two intervals (a_1, c_1) and (c_1, b_1) for some $c_1 \in (a_1, b_1)$ and by using the LC solutions on each interval constructed in [6] when a_1 and b_1 are singular in [7]. In [8], Suo and Wang extend the characterization in [7] to two-interval case but the result reduces to the case when one or two or three or four endpoints are regular.

As noted in [3], a simple way of getting self-adjoint operators in a direct sum Hilbert space is to take the direct sum of self-adjoint operators from each of the separate Hilbert spaces. However, there are many self-adjoint operators which are not merely the sum of self-adjoint operators from each of the separate intervals. These "new" self-adjoint operators involve interactions between the two intervals. Therefore in [3] the authors develop a "two-interval" theory. Mukhtarov and Yakubov[9] observe that the set of two-interval self-adjoint realizations can be significantly enlarged by using different multiples of the usual inner products associated with each of the intervals. In [10] and [11], Sun, Wang and Zettl use the Mukhtarov-Yakubov modification of the Everitt-Zettl theory to characterize all self-adjoint realizations of regular and singular two-interval Sturm-Liouville problems respectively in the modified direct sum Hilbert space with different inner product multiples. They generate self-adjoint realizations for the second order case with coupled boundary conditions and a real coupling matrix K, the method of [3] requires that $\det(K) = 1$ whereas with the Mukhtarov-Yakubov modification in [10] and [11] it is only required that $\det(K)$ is positive. In [12], Suo and Wang use the Mukhtarov-Yakubov modification to characterize all self-adjoint realizations of two-interval even order problems with one endpoint of each interval (a_1, b_1), (a_2, b_2) regular using Hilbert spaces but with the usual inner products replaced by appropriate multiples. The interplay of these multiples with the boundary conditions generates self-adjoint problems of even order with real coupling matrices K which are much more general than the coupling matrices from the $''$unmodified$''$ theory.

In this paper we characterize all self-adjoint realizations for singular two-interval even order problems in the modified direct sum Hilbert space with different inner product multiples. As in [12], the interplay of these multiples with the boundary conditions generates self-adjoint problems with real coupling matrices K which are much more general than the coupling matrices from the $''$unmodified$''$ theory. Our result reduces to the case when one or two or three or four endpoints are regular. Thus the results given in [8] are special cases of our results and can be obtained simply by taking both of the multiple inner product parameters to be unity, and the results given in [12] are special cases of our results and can be obtained when one endpoint of each interval $(a_1, b_1),\ (a_2, b_2)$ is regular. We give a number of examples to illustrate this additional generality, among other things.

From another perspective, instead of using multiples of the usual inner products, our approach can be described as using multiples of weight functions.

2 Notation and basic facts for one interval

Although we only consider even order equations with real coefficients in this paper, we summarize some basic facts about general quasi-differential equations of even and odd orders and real or complex coefficients for the convenience of the reader.

Let $J = (a, b)$ be an interval with $-\infty \leqslant a < b \leqslant \infty$ and let n be a positive integer (even or odd). For a given set S, $M_n(S)$ denotes the set of $n \times n$ complex matrices with entries from S.

Let

$$\begin{aligned} Z_n(J) := \{Q = (q_{is})_{i,s=1}^n, q_{i,i+1} &\neq 0 \text{ a.e. on } J,\ q_{i,i+1}^{-1} \in L_{loc}(J),\ 1 \leqslant i \leqslant n-1, \\ q_{is} &= 0 \text{ a.e. on } J,\ 2 \leqslant i+1 < s \leqslant n; \\ q_{is} &\in L_{loc}(J),\ s \neq i+1,\ 1 \leqslant i \leqslant n-1\}. \end{aligned} \tag{1}$$

Let $Q \in Z_n(J)$. We define

$$V_0 := \{y : J \to \mathbb{C},\ y \text{ is measurable}\} \tag{2}$$

and

$$y^{[0]} := y \qquad (y \in V_0). \tag{3}$$

Inductively, for $i = 1, \ldots, n$, we define

$$V_i = \{y \in V_{i-1} : y^{[i-1]} \in AC_{loc}(J)\}, \tag{4}$$

$$y^{[i]} = q_{i,i+1}^{-1}\Big\{y^{[i-1]\prime} - \sum_{s=1}^{i} q_{is} y^{[s-1]}\Big\} \quad (y \in V_i), \tag{5}$$

where $q_{n,n+1} := 1$, and $AC_{loc}(J)$ denotes the set of complex valued functions which are absolutely continuous on all compact subintervals of J. Finally we set

$$M\, y = M_Q\, y := i^n y^{[n]} \qquad (y \in V_n). \tag{6}$$

The expression $M = M_Q$ is called the quasi-differential expression associated with Q. For V_n we also use the notations $V(M)$ and $D(Q)$. The function $y^{[i]}$ $(0 \leqslant i \leqslant n)$ is called the i-th quasi-derivative of y. Since the quasi-derivative depends on Q, we sometimes write $y_Q^{[i]}$ instead of $y^{[i]}$.

Remark 2.1 The operator $M : D(Q) \to L_{loc}(J)$ is linear.

Let $Z_n(J, \mathbb{R})$ denote the matrices $Q \in Z_n(J)$ which have real valued components.

Definition 2.1 Let $Q \in Z_n(J, \mathbb{R})$ and let $M = M_Q$ be defined as above. Assume that

$$Q = -E^{-1}Q^*E, \text{ where } E = ((-1)^i \delta_{i,n+1-s})_{i,s=1}^n. \tag{7}$$

Then $M = M_Q$ is called a symmetric differential expression.

Let $w \in L_{loc}(J)$ be positive a.e. on J. We consider the Hilbert space

$$H = L^2(J, w)$$

with its usual inner product

$$\langle y, v\rangle_w := \int_J y\, \overline{v}\, w.$$

The maximal and minimal operators associated with a symmetric expression Q and a positive weight function w in the Hilbert space H are defined as follows.

Definition 2.2 Assume $Q \in Z_n(J, \mathbb{R})$ satisfies (7) and let $M = M_Q$ be the associated symmetric expression. Let $w \in L_{loc}(J)$ be positive a.e. on J. Define

$$D_{\max} = \{y \in L^2(J, w) : y \in D(Q),\ w^{-1}M\ y \in L^2(J, w)\},$$

$$S_{\max} y = w^{-1} M\ y, \qquad y \in D_{\max}.$$
$$S_{\min} = S_{\max}^*,$$
$$D_{\min} = D(S_{\min}). \tag{8}$$

Lemma 2.1 Let $S_{\min}$ and $S_{\max}$ be defined as above. Then $D_{\min}$ and $D_{\max}$ are dense in H, $S_{\min}$ and $S_{\max}$ are closed operators in H, $S_{\min}^* = S_{\max}$, $S_{\min} = S_{\max}^*$ and $S_{\min}$ is a symmetric operator in H.

From Definition 2.2 and Lemma 2.1 we see that every self-adjoint extension S of the minimal operator is 'between' the minimal and maximal operators, i.e., we have

$$S_{\min} \subset S = S^* \subset S_{\max}. \tag{9}$$

Thus these self-adjoint operators S are distinguished from one another only by their domains.

Lemma 2.2 [Lagrange Identity] Assume $Q \in Z_n(J, \mathbb{R})$ satisfies (7) and let $M = M_Q$ be the corresponding differential expression. Then for any $y,\ z \in D(Q)$ we have

$$\overline{z} My - y\,\overline{Mz} = [y, z]', \tag{10}$$

where

$$[y, z] = (-1)^k \sum_{i=0}^{n-1} (-1)^{n+1-i}\, \overline{z}^{[n-i-1]}\, y^{[i]} = (-1)^k (Z^* E Y), \tag{11}$$

$$Y = \begin{pmatrix} y \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix},\ Z = \begin{pmatrix} z \\ z^{[1]} \\ \vdots \\ z^{[n-1]} \end{pmatrix}. \tag{12}$$

Definition 2.3 [Regular Endpoints] Let $Q \in Z_n(J, \mathbb{R}),\ J = (a, b)$. The expression $M = M_Q$ is said to be regular at a if for some c, $a < c < b$, we have

$$q_{i,i+1}^{-1} \in L(a, c),\ i = 1, \cdots, n-1;$$
$$q_{is} \in L(a, c),\ 1 \leqslant i, s \leqslant n,\ s \neq i+1.$$

Similarly the endpoint b is regular if for some c, $a < c < b$, we have

$$q_{i,i+1}^{-1} \in L(c, b),\ i = 1, \cdots, n-1;$$
$$q_{is} \in L(c, b),\ 1 \leqslant i,\ s \leqslant n,\ s \neq i+1.$$

Note that, from (1) it follows that if the above hold for some $c \in J$ then they hold for any $c \in J$. We say that M is regular on J, or just M is regular, if M is regular at both endpoints.

3 Notation and basic assumptions for two-intervals

Let

$$J_r = (a_r, b_r), -\infty \leqslant a_r < b_r \leqslant \infty, r = 1, 2.$$

Define two differential expressions with real-valued coefficients by

$$M_r\, y = M_{Q_r}\, y := i^n y^{[n]} \text{ on } J_r, r = 1, 2, n = 2k,\ k > 1. \tag{13}$$

Let

$$H_r = L^2(J_r, w_r), w_r > 0, r = 1, 2.$$

The two-interval maximal and minimal domains and operators are simply the direct sums of the corresponding one-interval domains and operators

$$D_{\max} = D_{1\max} + D_{2\max},\ D_{\min} = D_{1\min} + D_{2\min}, \tag{14}$$

$$S_{\max} = S_{1\max} + S_{2\max},\ S_{\min} = S_{1\min} + S_{2\min}. \tag{15}$$

Elements of $H_u = H_1 + H_2$ will be denoted in bold face type: $\mathbf{f} = \{f_1, f_2\}$ with $f_1 \in H_1,\ f_2 \in H_2$. As usual the inner product in H_u is defined by

$$(\boldsymbol{f}, \boldsymbol{g}) = (f_1, g_1)_1 + (f_2, g_2)_2, \tag{16}$$

where $(\cdot,\cdot)_r$ is the usual inner pruduct in H_r:

$$(f_r, g_r)_r = \int_{J_r} f_r \bar{g}_r w_r. \tag{17}$$

In this paper, following [10] we replace the direct sum inner product (16) by

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = l(f_1, g_1)_1 + s(f_2, g_2)_2,\ l > 0,\ s > 0, \tag{18}$$

and apply operator theory in the direct sum space

$$H = (L^2(J_1, w_1) \dot{+} L^2(J_2, w_2),\ \langle\cdot,\cdot\rangle). \tag{19}$$

Remark 3.1 For any positive numbers l and s, (18) is an inner product in H. The underlying set of the Hilbert space H defined by (19) is the same as that of the usual direct sum Hilbert spaces H_u, thus these spaces are differentiated from each other only by their inner products. As we will see below the parameters l, s influence the boundary conditions which yield self-adjoint realizations of the equations in the two-interval case. From another perspective, we observe also that the Hilbert space (19) can be described as a $''$usual$''$ direct sum space H_u with summands $H_1 = L^2(J_1, lw_1)$,$H_2 = L^2(J_2, sw_2)$.

As in the one interval case the Lagrange sesquilinear form plays an important role. It is defined, for appropriate functions $\boldsymbol{f}, \boldsymbol{g}$, by

$$[\boldsymbol{f}, \boldsymbol{g}] = l\,[f_1, g_1]_1(b_1) - l\,[f_1, g_1]_1(a_1) + s\,[f_2, g_2]_2(b_2) - s\,[f_2, g_2]_2(a_2), \tag{20}$$

where

$$[f_r, g_r]_r = (-1)^k (G_r^* E F_r),\ F_r = \begin{pmatrix} f_r \\ f_r^{[1]} \\ \vdots \\ f_r^{[n-1]} \end{pmatrix},\ G_r = \begin{pmatrix} g_r \\ g_r^{[1]} \\ \vdots \\ g_r^{[n-1]} \end{pmatrix}. \tag{21}$$

Note that the two-interval Lagrange form $[f,g]$ connects all four endpoints with each other and depends on the parameters l,s.

4 Characterization of all self-adjoint domains for two-interval problems

In this section we assume that $M_{Q_r}(r=1,2)$ are generated by $Q_r \in Z_{n(r)}(J_r,\mathbb{R})$ satisfying (7), $n=2k$, $k>1$. First we give some preliminary lemmas.

Lemma 4.1 We have

1.

$$S_{\min}^{*}=S_{1\min}^{*}+S_{2\min}^{*}=S_{1\max}+S_{2\max}=S_{\max},$$

$$S_{\max}^{*}=S_{1\max}^{*}+S_{2\max}^{*}=S_{1\min}+S_{2\min}=S_{\min}.$$

In particular,

$$D_{\max}=D(S_{\max})=D(S_{1\max})+D(S_{2\max}),$$

$$D_{\min}=D(S_{\min})=D(S_{1\min})+D(S_{2\min}).$$

2. The minimal operator $S_{\min}$ is a closed, symmetric, densely defined operator in the Hilbert space H with deficiency index d given by $d=d_1+d_2$.

Proof The proof given in [3] for (16) extends readily to (18).

Definition 4.1 Assume that $a_r\leqslant\alpha_r<\beta_r\leqslant b_r$ and $S_{r\min}$ are defined on (α_r,β_r) as above. Then the deficiency indexes d_r of $S_{r\min}$ are the number of linearly independent solutions of

$$M_r\,y=i\,w_r\,y\ \text{ on }\ (\alpha_r,\beta_r),\ i=\sqrt{-1},r=1,2,$$

which lie in $L^2((\alpha_r,\beta_r),w_r)$.

Lemma 4.2 Let $a_r\leqslant\alpha_r<\beta_r\leqslant b_r$.The number d_r of linearly independent solutions of

$$M_r\,y=\lambda_r\,w_r\,y\ \text{ on }\ (\alpha_r,\beta_r) \tag{22}$$

lying in $L_2((\alpha_r,\beta_r),w_r)$ is independent of $\lambda_r\in\mathbb{C}$, provided $Im(\lambda_r)\neq 0$. If one endpoint of (α_r,β_r) is regular and the other is singular then the inequalities

$$k\leqslant d_r\leqslant 2k=n \tag{23}$$

hold. For $\lambda=\lambda_r\in\mathbb{R}$, the number of linearly independent solutions of $(22)_{r=1}$ lying in $L_2((\alpha_1,\beta_1),w_1)$ is less than or equal to d_1 and the number of linearly independent solutions of $(22)_{r=2}$ lying in $L_2((\alpha_2,\beta_2),w_2)$ is less than or equal to d_2.

Let $c_r\in(a_r,b_r)=J_r, r=1,2$.

a. If d_{11} is the deficiency index on (a_1,c_1), d_{12} is the deficiency index on (c_1,b_1) and d_1 is the deficiency index on (a_1,b_1), then

$$d_1=d_{11}+d_{12}-n. \tag{24}$$

b. If d_{21} is the deficiency index on (a_2,c_2), d_{22} is the deficiency index on (c_2,b_2) and d_2 is the deficiency index on (a_2,b_2), then

$$d_2=d_{21}+d_{22}-n. \tag{25}$$

Proof For the proof of the previous statement see Lemma 4 in [8]. The statement of a and b see Lemma 7 in [7].

Next we give the Everitt and Zettl[5] extension of the GKN Theorem from the one interval to the two-interval case.

Lemma 4.3 Let $S_{\min}$ be the two interval minimal operator in H and let d be the deficiency index of $S_{\min}$. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exist vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_d$ in $D_{\max}$ satisfying the following conditions:

(i) $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_d$ are linearly independent modulo $D_{\min}$;

(ii) $[\boldsymbol{w}_i, \boldsymbol{w}_j] = l[w_{i1}, w_{j1}]_1(b_1) - l[w_{i1}, w_{j1}]_1(a_1) + s[w_{i2}, w_{j2}]_2(b_2) - s[w_{i2}, w_{j2}]_2(a_2) = 0,\ i, j = 1, \cdots, d$;

(iii) $D(S) = \{\boldsymbol{y} = \{y_1, y_2\} \in D_{\max} : [\boldsymbol{y}, \boldsymbol{w}_j] = l[y_1, w_{j1}]_1(b_1) - l[y_1, w_{j1}]_1(a_1)$
$+s[y_2, w_{j2}]_2(b_2) - s[y_2, w_{j2}]_2(a_2) = 0,\ j = 1, \cdots, d\}$.

Proof See Theorem 3.1 and Corollary 3.3 in Everitt and Zettl[5] for the case with inner product (16), the adaptation to inner product (18) is routine.

Remark 4.1 As mentioned in the Introduction, the characterization of Lemma 4.3 depends on the maximal domain vectors $w_j,\ j = 1, \cdots, d$. These vectors depend on the coefficients of each differential equation and this dependence is implicit and complicated. Next we give more explicit equivalent conditions for (i)-(iii) of Lemma 4.3.

The next theorem is our main result in this paper.

Theorem 4.1 Let M_r be a symmetric differential expression on (a_r, b_r) and let $c_r \in (a_r, b_r)$. Let the Lagrange form $[\cdot, \cdot]$ be given by (20), $d_1 = d_{11} + d_{12} - n$, $d_2 = d_{21} + d_{22} - n$, $m_r = 2d_{1r} - n$ and $n_r = 2d_{2r} - n$. Then d_1 is the deficiency index of $(22)_{r=1}$ on (a_1, b_1) and d_2 is the deficiency index of $(22)_{r=2}$ on (a_2, b_2) and $d = d_1 + d_2$. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exists a complex $d \times m_1$ matrix A_1 and a complex $d \times m_2$ matrix B_1 and a complex $d \times n_1$ matrix A_2 and a complex $d \times n_2$ matrix B_2 such that the following three conditions hold:

1. $\operatorname{rank}(A_1, B_1, A_2, B_2) = d$;
2. $sA_1E_{m_1}A_1^* - sB_1E_{m_2}B_1^* + lA_2E_{n_1}A_2^* - lB_2E_{n_2}B_2^* = 0$;
3. $D(S) = \{y = \{y_1, y_2\} \in D_{\max}$:

$$A_1\begin{pmatrix}[y_1,u_1]_1(a_1)\\ \vdots\\ [y_1,u_{m_1}]_1(a_1)\end{pmatrix}+B_1\begin{pmatrix}[y_1,v_1]_1(b_1)\\ \vdots\\ [y_1,v_{m_2}]_1(b_1)\end{pmatrix}+$$
$$A_2\begin{pmatrix}[y_2,g_1]_2(a_2)\\ \vdots\\ [y_2,g_{n_1}]_2(a_2)\end{pmatrix}+B_2\begin{pmatrix}[y_2,z_1]_2(b_2)\\ \vdots\\ [y_2,z_{n_2}]_2(b_2)\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\}. \tag{26}$$

Here $u_1, \ldots, u_{m_1}$,$v_1, \ldots, v_{m_2}$, $g_1, \ldots, g_{n_1}$ and $z_1, \ldots, z_{n_2}$ are solutions of LC type on (a_1, c_1), (c_1, b_1), (a_2, c_2) and (c_2, b_2), respectively, as constructed in [6] and E_j is the symplectic matrix (7) of order j. For details of the definition and construction of LC solutions see [6].

Proof The proof consists of applying Theorem 4.12 in [8] with the weight functions w_1, w_2 replaced by $w_1/l, w_2/s$, respectively. Note that $\operatorname{rank}(A_1, B_1, A_2, B_2) = \operatorname{rank}(lA_1, lB_1, sA_2, sB_2)$ for any $l > 0, s > 0$. Proceeding as in [8] we note that the introduction of the parameters l, s into the weight functions results in the Lagrange form (20) in place of $[\boldsymbol{f}, \boldsymbol{g}] = [f_1, g_1]_1(b_1) - [f_1, g_1]_1(a_1) + [f_2, g_2]_2(b_2) - [f_2, g_2]_2(a_2)$.

The proof of Theorem 4.1 is completed by choosing

$$A_1 = -l(\overline{a}_{ij})_{d\times m_1}, B_1 = l(\overline{b}_{ij})_{d\times m_2}, A_2 = -s(\overline{c}_{ij})_{d\times n_1}, B_2 = s(\overline{d}_{ij})_{d\times n_2}.$$

See also the proof of Theorem 9 of [7] or Theorem 4 of [6]. They are the one-interval case and reveal the basic strategy.

In Theorem 4.1, it is assumed that all four endpoints a_1, b_1, a_2, b_2 are singular. It can be specialized to known results when one or two or three or four endpoints are regular. Theorem 4.1 reduces to Theorem 2 in [12] when one endpoint of each interval (a_1, b_1), (a_2, b_2) is regular. We have either a_1 is regular, b_1 is singular, a_2 is regular, b_2 is singular or a_1 is singular, b_1 is regular, a_2 is singular, b_2 is regular, or a_1 is singular, b_1 is regular, a_2 is regular, b_2 is singular, or a_1 is regular, b_1 is singular, a_2 is singular, b_2 is regular. We state one case here for the convenience of the reader.

Theorem 4.2 Let the hypotheses and notation of Theorem 4.1 hold and assume that a_1 and a_2 are regular. Then $d = d_{12} + d_{22}$. The solutions $v_1, v_2, \cdots, v_{d_{12}}$ and $z_1, z_2, \cdots, z_{d_{22}}$ can be extended to solutions on (a_1, b_1) and (a_2, b_2) such that $v_1, v_2, \cdots, v_{d_{12}} \in L^2((a_1, b_1), w_1)$ and $z_1, z_2, \cdots, z_{d_{22}} \in L^2((a_2, b_2), w_2)$. Let $m_2 = 2d_{12} - n$ and $n_2 = 2d_{22} - n$. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exists a complex $d \times n$ matrix A_1 and a complex $d \times m_2$ matrix B_1 and a complex $d \times n$ matrix A_2 and a complex $d \times n_2$ matrix B_2 such that the following three conditions hold:

1. $\mathrm{rank}(A_1, B_1, A_2, B_2) = d$;
2. $sA_1E_nA_1^* - sB_1E_{m_2}B_1^* + lA_2E_nA_2^* - lB_2E_{n_2}B_2^* = 0$;
3. $D(S) = \{y = \{y_1, y_2\} \in D_{\max}$:

$$A_1\begin{pmatrix} y_1(a_1) \\ \vdots \\ y_1^{[n-1]}(a_1)\end{pmatrix} + B_1\begin{pmatrix} [y_1, v_1]_1(b_1) \\ \vdots \\ [y_1, v_{m_2}]_1(b_1)\end{pmatrix} +$$

$$A_2\begin{pmatrix} y_2(a_2) \\ \vdots \\ y_2^{[n-1]}(a_2)\end{pmatrix} + B_2\begin{pmatrix} [y_2, z_1]_2(b_2) \\ \vdots \\ [y_2, z_{n_2}]_2(b_2)\end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}\}. \tag{27}$$

Proof The proof is a combination of Theorem 4.14 in [8] and Theorem 2 in [12].

5 Examples

To illustrate the self-adjoint boundary conditions given by Theorem 4.1 we give a number of examples. Here we give examples for

$$n = 4,\ 0 \leqslant d \leqslant 8.$$

Similar examples can be easily constructed for all higher order cases $n = 2k,\ k > 2$. Since the conditions when $d = 0$ or 1 are the same as in the one interval case and are independent of l and s. In these cases the self-adjoint extensions in the Hilbert space H defined by (18) are the same as those of the usual direct sum Hilbert space H_u defined by (16). So we give examples here only for $d = 2, d = 3, d = 4, d = 5, d = 6, d = 7$ and $d = 8$.

Example 5.1 Assume $d_{11} = 2, d_{12} = 3, d_{21} = 3, d_{22} = 2$. Then $d_1 = d_{11}+d_{12}-4 = 1,\ d_2 = d_{21}+d_{22}-4 = 1, d = d_1 + d_2 = 2$ and $m_1 = 2d_{11} - 4 = 0,\ m_2 = 2d_{12} - 4 = 2,\ n_1 = 2d_{21} - 4 = 2,\ n_2 = 2d_{22} - 4 = 0$. Suppose that the boundary conditions at b_1, a_2 are coupled:

$$\begin{pmatrix} [y_1, v_1]_1(b_1) \\ [y_1, v_2]_1(b_1) \end{pmatrix} = K \begin{pmatrix} [y_2, g_1]_2(a_2) \\ [y_2, g_2]_2(a_2) \end{pmatrix}, \tag{28}$$

$$K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2, \det(K) > 0.$$

Let

$$B_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ A_2 = K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}.$$

Then $\mathrm{rank}(B_1, A_2) = 2$. From a straightforward computation, it follows that

$sB_1E_2B_1^* = lA_2E_2A_2^*$ is equivalent with $s = l\det(K)$.

By Theorem 4.1 we have if $l = 1$, and $\det(K) = s > 0$, then the boundary conditions (28) are self-adjoint.

Remark 5.1 Note that $s > 0$ is needed to preserve the positivity of the inner product (18). Using appropriate multiples of the usual inner product, or changing the weight function w_2 to sw_2 we can generate self-adjoint operators for any real coupling matrix K satisfying $\det(K) = s > 0$. This contrasts with the results in [8], using the weight function w_2 which requires that $\det(K) = 1$ for self-adjointness. We see that the parameter s plays a role in establishing the self-adjoint boundary conditions.

Example 5.2 Assume $d_{11} = 2, d_{12} = 3, d_{21} = 3, d_{22} = 3$. Then $d_1 = 1,\ d_2 = 2, d = 3$ and $m_1 = 0,\ m_2 = 2,\ n_1 = 2,\ n_2 = 2$. Consider separated condition at a_2 and coupled conditions at b_1, b_2:

$$C_1[y_2, g_1]_2(a_2) + C_2[y_2, g_2]_2(a_2) = 0,\ C_1, C_2 \in R,\ (C_1, C_2) \neq (0, 0),$$

$$\begin{pmatrix} [y_1, v_1]_1(b_1) \\ [y_1, v_2]_1(b_1) \end{pmatrix} = K \begin{pmatrix} [y_2, z_1]_2(b_2) \\ [y_2, z_2]_2(b_2) \end{pmatrix}, \tag{29}$$

$$K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ i, j = 1, 2, \det(K) < 0.$$

Let

$$B_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ c_1 & c_2 \end{pmatrix}, B_2 = K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ 0 & 0 \end{pmatrix}.$$

In this case $\mathrm{rank}(B_1, A_2, B_2) = 3$ and $A_2E_2A_2^* = 0$. Then in terms of Theorem , we obtain the equivalence of the conditions for self-adjointness:

$$-s = l\det(K).$$

So, if $l = 1$, and $\det(K) = -s < 0$ then boundary conditions (29) are self-adjoint.

Note that by studying the two-interval theory in direct sum spaces with inner product multiples we obtain self-adjoint operators for any real coupling matrix K satisfying $\det(K) = -s < 0$. This contrasts with the results in [8] which require $\det(K) = -1$.

Example 5.3 Assume $d_{11}=3, d_{12}=3, d_{21}=3, d_{22}=3$. Then $d_1=2,\ d_2=2, d=4$ and $m_1=2,\ m_2=2,\ n_1=2,\ n_2=2$. Consider two pairs of coupled conditions:

$$\begin{pmatrix}[y_2,g_1]_2(a_2)\\ [y_2,g_2]_2(a_2)\end{pmatrix}=G\begin{pmatrix}[y_1,v_1]_1(b_1)\\ [y_1,v_2]_1(b_1)\end{pmatrix}, \tag{30}$$

$$G=(g_{ij}),\ g_{ij}\in\mathbb{R}, i,j=1,2, \det(G)>0,$$

$$\begin{pmatrix}[y_2,z_1]_2(b_2)\\ [y_2,z_2]_2(b_2)\end{pmatrix}=K\begin{pmatrix}[y_1,u_1]_1(a_1)\\ [y_1,u_2]_1(a_1)\end{pmatrix}, \tag{31}$$

$$K=(k_{ij}),\ k_{ij}\in\mathbb{R}, i,j=1,2, \det(K)>0.$$

Proceeding as in the previous example we obtain the equivalence of conditions for self-adjointness:

$$sGE_2G^*=lE_2 \quad \text{and} \quad sKE_2K^*=lE_2,$$

$$s\det(G)=l \quad \text{and} \quad s\det(K)=l,$$

i.e.,

$$\det(G)=\det(K)=\frac{l}{s}.$$

This shows that (30) and (31) are self-adjoint boundary conditions when positive numbers l, s satisfy $\det(G)=\det(K)=l/s$.

Example 5.4 Assume $d_{11}=2, d_{12}=4, d_{21}=4, d_{22}=3$. Then $d_1=2,\ d_2=3, d=5$ and $m_1=0,\ m_2=4,\ n_1=4,\ n_2=2$. Consider separated conditions at b_2 and coupled conditions at b_1, a_2:

$$C_1[y_2,z_1]_2(b_2)+C_2[y_2,z_2]_2(b_2)=0,\ C_1,C_2\in R,\ (C_1,C_2)\neq(0,0),$$

$$\begin{pmatrix}[y_1,v_1]_1(b_1)\\ [y_1,v_2]_1(b_1)\\ [y_1,v_3]_1(b_1)\\ [y_1,v_4]_1(b_1)\end{pmatrix}=K\begin{pmatrix}[y_2,g_1]_2(a_2)\\ [y_2,g_2]_2(a_2)\\ [y_2,g_3]_2(a_2)\\ [y_2,g_4]_2(a_2)\end{pmatrix}. \tag{32}$$

$$K=(k_{ij}),\ k_{ij}\in\mathbb{R}, i,j=1,2,3,4, M_{14}-N_{14}<0, M_{23}-N_{23}>0,$$

$$M_{ij}=\begin{vmatrix}k_{i2} & k_{i3}\\ k_{j2} & k_{j3}\end{vmatrix}, N_{ij}=\begin{vmatrix}k_{i1} & k_{i4}\\ k_{j1} & k_{j4}\end{vmatrix}, i<j, i=1,2,3, j=2,3,4.$$

Let

$$B_1=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\\0&0&0&0\end{pmatrix}, A_2=K=\begin{pmatrix}k_{11}&k_{12}&k_{13}&k_{14}\\k_{21}&k_{22}&k_{23}&k_{24}\\k_{31}&k_{32}&k_{33}&k_{34}\\k_{41}&k_{42}&k_{43}&k_{44}\\0&0&0&0\end{pmatrix}, B_2=\begin{pmatrix}0&0\\0&0\\0&0\\0&0\\c_1&c_2\end{pmatrix}.$$

In this case $\operatorname{rank}(B_1,A_2,B_2)=5$ and $B_2E_2B_2^*=0$. Then in terms of Theorem 4.1, we obtain the equivalence of the conditions for self-adjointness:

1. $M_{12}=N_{12}$; 2. $M_{13}=N_{13}$; 3. $M_{24}=N_{24}$;

4. $M_{34}=N_{34}$; 5. $l(M_{14}-N_{14})=-s$; 6. $l(M_{23}-N_{23})=s$.

If we choose $l=1$, and $s>0$ satisfies $M_{14}-N_{14}=-s$, $M_{23}-N_{23}=s$ and (1),(2),(3),(4), then boundary conditions (32) are self-adjoint.

Remark 5.2 Using appropriate multiples of the usual inner product, we establish self-adjoint operators for any real coupling matrix K satisfying $M_{14} - N_{14} = -s < 0$, $M_{23} - N_{23} = s > 0$ and (1), (2), (3), (4). This contrasts with the results in [8] which requires $M_{14} - N_{14} = -1$, $M_{23} - N_{23} = 1$ and (1), (2), (3), (4).

Example 5.5 Assume $d_{11} = 4, d_{12} = 3, d_{21} = 3, d_{22} = 4$. Then $d_1 = 3,\ d_2 = 3, d = 6$ and $m_1 = 4,\ m_2 = 2, n_1 = 2, n_2 = 4$. Consider two pairs of coupled conditions:

$$\begin{pmatrix} [y_2, g_1]_2(a_2) \\ [y_2, g_2]_2(a_2) \end{pmatrix} = G \begin{pmatrix} [y_1, v_1]_1(b_1) \\ [y_1, v_2]_1(b_1) \end{pmatrix}, \tag{33}$$

$$G = (g_{ij}),\ g_{ij} \in \mathbb{R}, i, j = 1, 2, \det(G) > 0,$$

$$\begin{pmatrix} [y_2, z_1]_2(b_2) \\ [y_2, z_2]_2(b_2) \\ [y_2, z_3]_2(b_2) \\ [y_2, z_4]_2(b_2) \end{pmatrix} = K \begin{pmatrix} [y_1, u_1]_1(a_1) \\ [y_1, u_2]_1(a_1) \\ [y_1, u_3]_1(a_1) \\ [y_1, u_4]_1(a_1) \end{pmatrix}. \tag{34}$$

$$K = (k_{ij}),\ k_{ij} \in \mathbb{R}, i, j = 1, 2, 3, 4, M_{14} - N_{14} < 0, M_{23} - N_{23} > 0,$$

$$M_{ij} = \begin{vmatrix} k_{i2} & k_{i3} \\ k_{j2} & k_{j3} \end{vmatrix}, N_{ij} = \begin{vmatrix} k_{i1} & k_{i4} \\ k_{j1} & k_{j4} \end{vmatrix}, i < j, i = 1, 2, 3, j = 2, 3, 4.$$

Proceeding as in the previous example we obtain the equivalence of conditions for self-adjointness:

$$sGE_2G^* = lE_2 \quad \text{and} \quad sKE_4K^* = lE_4,$$

$s\det(G) = l$ and

1. $M_{12} = N_{12}$; 2. $M_{13} = N_{13}$; 3. $M_{24} = N_{24}$;

4. $M_{34} = N_{34}$; 5. $s(M_{14} - N_{14}) = -l$; 6. $s(M_{23} - N_{23}) = l$.

If we choose $s = 1$, and $l > 0$ satisfies $\det(G) = l > 0$ and $M_{14} - N_{14} = -l < 0$, $M_{23} - N_{23} = l > 0$ and (1), (2), (3), (4), then the boundary conditions (33) and (34) are self-adjoint.

Remark 5.3 Using appropriate multiples of the usual inner product, we establish self-adjoint operators for any real coupling matrix K satisfying $\det(G) = l > 0$ and $M_{14} - N_{14} = -l < 0$, $M_{23} - N_{23} = l > 0$ and (1), (2), (3), (4). This contrasts with the results in [8] which requires $\det(G) = 1$ and $M_{14} - N_{14} = -1$, $M_{23} - N_{23} = 1$ and (1), (2), (3), (4).

Example 5.6 Assume $d_{11} = 3,\ d_{12} = 4,\ d_{21} = 4,\ d_{22} = 4$. Then $d_1 = 3,\ d_2 = 4, d = 7$ and $m_1 = 2,\ m_2 = 4,\ n_1 = 4,\ n_2 = 4$. Consider separated conditions at a_1 and at b_1 and coupled conditions at $a_2,\ b_2$:

$$C_1[y_1, u_1]_1(a_1) + C_2[y_1, u_2]_1(a_1) = 0,\ C_1, C_2 \in R,\ (C_1, C_2) \neq (0, 0),$$

$$[y_1, v_1]_1(b_1) + i[y_1, v_2]_1(b_1) = 0, i[y_1, v_3]_1(b_1) + [y_1, v_4]_1(b_1) = 0, \tag{35}$$

$$A_2 \begin{pmatrix} [y_2, g_1]_2(a_2) \\ [y_2, g_2]_2(a_2) \\ [y_2, g_3]_2(a_2) \\ [y_2, g_4]_2(a_2) \end{pmatrix} + B_2 \begin{pmatrix} [y_2, z_1]_2(b_2) \\ [y_2, z_2]_2(b_2) \\ [y_2, z_3]_2(b_2) \\ [y_2, z_4]_2(b_2) \end{pmatrix} = 0. \tag{36}$$

Then $sA_1E_2A_1^* - sB_1E_4B_1^* = 0$ for any s since $A_1E_2A_1^* = 0 = B_1E_4B_1^*$. In terms of Theorem , the boundary conditions (35) and (36) are self-adjoint if and only if $\text{rank}(A_2, B_2) = 4$ and

$$A_2E_4A_2^* - B_2E_4B_2^* = 0.$$

Note that these conditions are independent of l and s and are simply the one-interval self-adjointness conditions for each of the two intervals separately. Thus the above example just gives the two-interval self-adjointness conditions which are generated by the direct sum of self-adjoint operators from each of the two intervals separately.

The next example involves interactions between endpoints of the two intervals and thus generates a self-adjoint operator which is not a direct sum of operators from each of the two intervals.

Example 5.7 In the maximal deficiency case $d = 8$, we have $d_{11} = 4,\ d_{12} = 4,\ d_{21} = 4,\ d_{22} = 4,\ d_1 = 4,\ d_2 = 4$, and $m_1 = 4,\ m_2 = 4,\ n_1 = 4,\ n_2 = 4$. Consider two pairs of coupled conditions:

$$\begin{pmatrix} [y_1,u_1]_1(a_1) \\ [y_1,u_2]_1(a_1) \\ [y_1,u_3]_1(a_1) \\ [y_1,u_4]_1(a_1) \end{pmatrix} = G \begin{pmatrix} [y_2,g_1]_2(a_2) \\ [y_2,g_2]_2(a_2) \\ [y_2,g_3]_2(a_2) \\ [y_2,g_4]_2(a_2) \end{pmatrix}, \tag{37}$$

$$G = (g_{ij}),\ g_{ij} \in \mathbb{R},\ i,j = 1,2,3,4,\ Q_{14} - R_{14} > 0,\ Q_{23} - R_{23} < 0,$$

$$Q_{ij} = \begin{vmatrix} g_{i2} & g_{i3} \\ g_{j2} & g_{j3} \end{vmatrix},\ R_{ij} = \begin{vmatrix} g_{i1} & g_{i4} \\ g_{j1} & g_{j4} \end{vmatrix}, i<j, i=1,2,3, j=2,3,4.$$

$$\begin{pmatrix} [y_1,v_1]_1(b_1) \\ [y_1,v_2]_1(b_1) \\ [y_1,v_3]_1(b_1) \\ [y_1,v_4]_1(b_1) \end{pmatrix} = K \begin{pmatrix} [y_2,z_1]_2(b_2) \\ [y_2,z_2]_2(b_2) \\ [y_2,z_3]_2(b_2) \\ [y_2,z_4]_2(b_2) \end{pmatrix} \tag{38}$$

$$K = (k_{ij}),\ k_{ij} \in \mathbb{R}, i,j = 1,2,3,4, M_{14} - N_{14} > 0, M_{23} - N_{23} < 0,$$

$$M_{ij} = \begin{vmatrix} k_{i2} & k_{i3} \\ k_{j2} & k_{j3} \end{vmatrix}, N_{ij} = \begin{vmatrix} k_{i1} & k_{i4} \\ k_{j1} & k_{j4} \end{vmatrix}, i<j, i=1,2,3, j=2,3,4.$$

Proceeding as in the previous example we obtain the equivalence of conditions for self-adjointness:

$$lGE_4G^* = -sE_4 \quad \text{and} \quad lKE_4K^* = -sE_4,$$

1. $Q_{12} = R_{12}$; 2. $Q_{13} = R_{13}$; 3. $Q_{24} = R_{24}$;
4. $Q_{34} = R_{34}$; 5. $l(Q_{14} - R_{14}) = s$; 6. $l(Q_{23} - R_{23}) = -s$.

and

(i) $M_{12} = N_{12}$; **(ii)** $M_{13} = N_{13}$; **(iii)** $M_{24} = N_{24}$;
(iv) $M_{34} = N_{34}$; **(v)** $l(M_{14} - N_{14}) = s$; **(vi)** $l(M_{23} - N_{23}) = -s$.

This shows that (37) and (38) are self-adjoint boundary conditions when positive numbers l, s satisfy $Q_{14} - R_{14} = R_{23} - Q_{23} = M_{14} - N_{14} = N_{23} - M_{23} = s/l$ and (1), (2), (3), (4), (i), (ii), (iii), (iv) hold.

References:

[1] Boyd J P. Sturm-liouville eigenvalue problems with an interior pole [J]. J Math Physics, 1981, 22(8):1575-1590.

[2] Gesztesy F, Kirsch W. One-Dimensional schrödinger operators with interactions singular on a discrete set [J]. J Reine u Ang Mathematic, 1985, 362:28-50.

[3] Everitt W N, Zettl A. Sturm-liouville differential operators in direct sum spaces [J]. Rocky Mountain J Math, 1986, 16(16): 497-516.

[4] Zettl A. Sturm-liouville theory [M]. RI: American Mathematical Society, 2005.

[5] Everitt W N, Zettl A. Differential operators generated by a countable number of quasi-differential expressions on the line [J]. Proc London Math Soc, 1992, 64(3):524-544.

[6] Wang A P, Sun J, Zettl A. Characterization of domains of self-adjoint ordinary differential operators [J]. Journal of Differential Equations, 2009, 246:1600-1622.

[7] Hao X L, Sun J, Wang A P, et al. Characterization of domains of self-adjoint ordinary differential operators II [J]. Results in Mathematics, 2012, 61:255-281.

[8] Suo J Q, Wang W Y. Two-interval even order differential operators in direct sum spaces [J]. Results in Mathematics, 2012, 62:13-32.

[9] Mukhtarov O S, Yakubov S. Problems for differential equations with transmission conditions [J]. Applicable Analysis, 2002, 81:1033-1064.

[10] Wang A P, Sun J, Zettl A. Two-interval sturm-liouville operators in modified hilbert spaces [J]. J Math Anal Appl, 2007, 328:390-399.

[11] Sun J, Wang A P, Zettl A. Two-interval sturm-liouville operators in direct sum spaces with inner product multiples [J]. Results in Mathematics, 2007, 50:155-168.

[12] Suo J Q, Wang W Y. Two-interval even order Differential Operators in modified Hilbert Spaces [J]. Bulletin of the Australian Mathematical Society, 2012, 85:241-260.

(责任编辑：包震宇)