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Rational quadratic trigonometric Bézier curve based on new basis with exponential functions

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Abstract: We construct a rational quadratic trigonometric Bézier curve with four shape parameters by introducing two exponential functions into the trigonometric basis functions in this paper. It has the similar properties as the rational quadratic Bézier curve. For given control points, the shape of the curve can be flexibly adjusted by changing the shape parameters and the weight. Some conics can be exactly represented when the control points, the shape parameters and the weight are chosen appropriately. The C^0, C^1 and C^2 continuous conditions for joining two constructed curves are discussed. Some examples are given.

Key words: quadratic trigonometric basis functions; rational quadratic trigonometric Bézier curve; shape parameters; exponential functions

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基于新指数基函数的有理二次三角 Bézier 曲线

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摘　要：通过在三角基函数中引入两个指数函数，构造了一种具有 4 个形状参数的有理二次三角 Bézier 曲线，它与有理二次 Bézier 曲线有着相类似的性质．给定控制顶点，该曲线可通过改变形状参数和权因子而调整形状．适当选取控制顶点、形状参数和权因子时，一些二次曲线可以被精确地表示．讨论了连接两条曲线所满足 C^0, C^1 和 C^2 的连续条件，并给出了一些例子．

关键词：二次三角基函数；有理二次三角 Bézier 曲线；形状参数；指数函数

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1 Introduction

In geometric modeling and computer graphics, for efficiently improving the shape of the curves, some methods of generating curves were presented by incorporating shape parameters, trigonometric or exponential functions into the classical basis functions (cf. [1-8]). In the past ten years, Bézier form of trigonometric curves with shape parameters has received very much attention in CAGD through to the efforts of many researchers. For example, Han proposed quadratic trigonometric Bézier curves and cubic trigonometric Bézier curves with a shape parameter in [9] and [10] respectively. Han et al introduced the cubic trigonometric Bézier curve with two shape parameters in [11]. Sheng et al constructed the quasi-quartic Bézier-type curves with a shape parameter in [12]. Bashir et al gave a class of quasi-quintic trigonometric Bézier curve with two shape parameters in [13]. Hussain et al presented the C^1 rational quadratic trigonometric polynomial spline with two shape parameters in [14].

We define a new rational quadratic trigonometric Bézier curve with four shape parameters by introducing two exponential functions into the basis functions in this paper. It is more flexible to control the shape than the presented curve in [15]. The constructed curve inherits most geometric properties of the rational quadratic Bézier curve. It can be used to exactly represent some conics like ellipses, circles and parabolas. The condition of C^2 continuity between two constructed curves is discussed. Some examples illustrate that the constructed curve in this paper provides an effective method for designing curves and geometric modeling.

The structure of this paper is as follows. Section 2 defines the quadratic trigonometric Bézier basis functions with four shape parameters. Section 3 constructs the rational quadratic trigonometric Bézier curve with four shape parameters. Section 4 gives the effect of the shape of the curve by the shape parameters and the weight. Section 5 discusses the composite constructed curves. Section 6 is the conclusion.

2 Quadratic trigonometric Bézier basis functions with four shape parameters

By introducing two exponential functions into the basis functions, we define a kind of quadratic trigonometric basis functions with four shape parameters and discuss some properties.

Definition 1 Let $\alpha, \beta \in [-1, 1]$, $\lambda, \mu \in [0, +\infty)$, for $t \in [0, 1]$, the following three functions are defined to be the new quadratic trigonometric Bézier basis functions with four shape parameters α, β, λ and μ:

$$\begin{cases} b_0(t) = \left(1 - \sin \frac{\pi}{2}t\right)\left(1 - \alpha \sin \frac{\pi}{2}t\right)\mathrm{e}^{-\lambda t} \\ b_1(t) = 1 - b_0(t) - b_2(t) \\ b_2(t) = \left(1 - \cos \frac{\pi}{2}t\right)\left(1 - \beta \cos \frac{\pi}{2}t\right)\mathrm{e}^{-\mu(1-t)} \end{cases}. \tag{1}$$

Theorem 1 The basis functions (1) have the following properties:

(i) Nonnegative property: $b_i(t) \geqslant 0$ for $i = 0, 1, 2$.

(ii) Partition of unity: $\sum\limits_{i=0}^{2} b_i(t) \equiv 1$.

(iii) Symmetry: $b_i(t; \alpha, \beta, \lambda, \mu) = b_{2-i}(1 - t; \beta, \alpha, \mu, \lambda)$ for $i = 0, 1, 2$.

(iv) Properties at the endpoints:

$$b_0(0) = 1,\ b_1(0) = 0,\ b_2(0) = 0,$$

$$b_0(1)=0,\ b_1(1)=0,\ b_2(1)=1,$$

$$b_0'(0)=-\frac{\pi}{2}(1+\alpha)-\lambda,\ b_1'(0)=\frac{\pi}{2}(1+\alpha)+\lambda,\ b_2'(0)=0,$$

$$b_0'(1)=0,\ b_1'(1)=-\frac{\pi}{2}(1+\beta)-\mu,\ b_2'(1)=\frac{\pi}{2}(1+\beta)+\mu,$$

$$b_0''(0)=\frac{\pi^2}{2}\alpha+\pi\alpha\lambda+\pi\lambda+\lambda^2,\ b_1''(0)=-\frac{1}{4}\mathrm{e}^{-\mu}\pi^2(1-\alpha)-\frac{\pi^2}{2}\alpha-\pi\alpha\lambda-\pi\lambda-\lambda^2,$$

$$b_2''(0)=\frac{1}{4}\mathrm{e}^{-\mu}\pi^2(1-\alpha),\ b_0''(1)=\frac{1}{4}\mathrm{e}^{-\lambda}\pi^2(1-\alpha),$$

$$b_1''(1)=-\frac{1}{4}\mathrm{e}^{-\lambda}\pi^2(1-\alpha)-\frac{\pi^2}{2}\beta-\pi\beta\mu-\pi\mu-\mu^2,\ b_2''(1)=\frac{\pi^2}{2}\beta+\pi\beta\mu+\pi\mu+\mu^2.$$

(v) Monotonicity: For fixed $t\in[0,1]$, $b_0(t)$ is monotonically decreasing for shape parameters α and λ; $b_2(t)$ is monotonically decreasing for shape parameters β and μ; $b_1(t)$ is monotonically increasing respectively for shape parameters α, β, λ and μ.

Proof The results can be obtained immediately from the definition of the basis functions in (1).

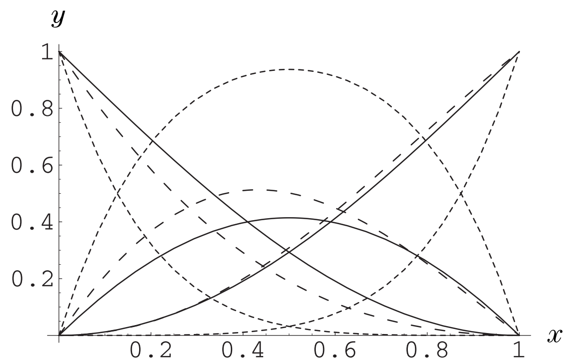

Figure 1 The new quadratic trigonometric Bézier basis functions

Figure 1 shows the curves of the quadratic trigonometric Bézier basis functions for $\alpha=\beta=\lambda=\mu=0$(solid lines), for $\alpha=\beta=1,\lambda=\mu=2$(short dashed lines), and for $\alpha=\frac{1}{2},\beta=-\frac{1}{2},\lambda=0,\mu=\frac{1}{2}$(long dashed lines).

3 Rational quadratic trigonometric Bézier curve with four shape parameters

We define a rational quadratic trigonometric Bézier curve with four shape parameters α,β,λ and μ as follows.

Definition 2 Suppose that we are given three control points $\boldsymbol{Q}_{\mathbf{i}}(i=0,1,2)$ in $\mathbb{R}^2$. The following function

$$r(t)=\frac{b_0(t)\boldsymbol{Q}_{\mathbf{0}}+\omega b_1(t)\boldsymbol{Q}_{\mathbf{1}}+b_2(t)\boldsymbol{Q}_{\mathbf{2}}}{b_0(t)+\omega b_1(t)+b_2(t)} \tag{2}$$

is called the rational quadratic trigonometric Bézier curve (RQTBC, for short) with four shape parameters, where the weight $\omega>0$, and the basis functions $b_i(t)(i=0,1,2)$ are defined in (1).

Some properties of the RQTBC can be obtained easily from the properties of the basis functions (1).

Theorem 2 The RQTBC (2) holds the following properties:

(i) Terminal properties:

$$r(0)=\boldsymbol{Q_0},\ r(1)=\boldsymbol{Q_2}, r'(0)=\Big(\frac{\pi}{2}+\frac{\pi}{2}\alpha+\lambda\Big)\omega(\boldsymbol{Q_1}-\boldsymbol{Q_0}), r'(1)=\Big(\frac{\pi}{2}+\frac{\pi}{2}\beta+\mu\Big)\omega(\boldsymbol{Q_2}-\boldsymbol{Q_1}),$$

$$r''(0)=\frac{\pi^2}{4}\mathrm{e}^{-\mu}(1-\beta)(\boldsymbol{Q_2}-\boldsymbol{Q_0})+\Big[\Big(\frac{\pi^2}{4}\mathrm{e}^{-\mu}(1-\beta)-\Big(\frac{\pi^2}{2}+\frac{\pi^2}{2}\alpha+\frac{\pi^2}{2}\alpha^2+\pi\lambda+\pi\alpha\lambda+\lambda^2\Big)\omega\Big)$$
$$+2\Big(\frac{\pi}{2}+\frac{\pi}{2}\alpha+\lambda\Big)^2\omega^2\Big](\boldsymbol{Q_0}-\boldsymbol{Q_1}),$$

$$r''(1)=\frac{\pi^2}{4}\mathrm{e}^{-\lambda}(1-\alpha)(\boldsymbol{Q_0}-\boldsymbol{Q_2})+\Big[\Big(\frac{\pi^2}{4}\mathrm{e}^{-\lambda}(1-\alpha)-\Big(\frac{\pi^2}{2}+\frac{\pi^2}{2}\beta+\frac{\pi^2}{2}\beta^2+\pi\mu+\pi\beta\mu+\mu^2\Big)\omega\Big)$$
$$+2\Big(\frac{\pi}{2}+\frac{\pi}{2}\beta+\mu\Big)^2\omega^2\Big](\boldsymbol{Q_2}-\boldsymbol{Q_1}).$$

(ii) Symmetry: if the weight ω is kept fixed, $\boldsymbol{Q_0},\boldsymbol{Q_1},\boldsymbol{Q_2}$ and $\boldsymbol{Q_2},\boldsymbol{Q_1},\boldsymbol{Q_0}$ define the same RQTBC in different parameterizations, i.e.,

$$r(t;\alpha,\beta,\lambda,\mu;\boldsymbol{Q_0},\boldsymbol{Q_1},\boldsymbol{Q_2})=r(1-t;\beta,\alpha,\mu,\lambda;\boldsymbol{Q_2},\boldsymbol{Q_1},\boldsymbol{Q_0}),$$

where $t\in[0,1]$, the shape parameters $\alpha,\beta\in[-1,1]$ and $\lambda,\mu\in[0,+\infty)$.

(iii) Geometric invariance: The shape of a RQTBC is independent of the choice of coordinates, i.e., it satisfies the following two equations:

$$R(t;\alpha,\beta,\lambda,\mu;\boldsymbol{Q_0}+\boldsymbol{q},\boldsymbol{Q_1}+\boldsymbol{q},\boldsymbol{Q_2}+\boldsymbol{q})=R(t;\alpha,\beta,\lambda,\mu;\boldsymbol{Q_0},\boldsymbol{Q_1},\boldsymbol{Q_2})+\boldsymbol{q},$$
$$R(t;\alpha,\beta,\lambda,\mu;\boldsymbol{Q_0}*\boldsymbol{T},\boldsymbol{Q_1}*\boldsymbol{T},\boldsymbol{Q_2}*\boldsymbol{T})=R(t;\alpha,\beta,\lambda,\mu;\boldsymbol{Q_0},\boldsymbol{Q_1},\boldsymbol{Q_2})*\boldsymbol{T},$$

where $\boldsymbol{q}$ is an arbitrary vector in $\mathbb{R}^2$, and $\boldsymbol{T}$ is an arbitrary 2×2 matrix.

(iv) Convex hull property: The entire RQTBC segment must lie inside its control polygon spanned by $\boldsymbol{Q_0},\boldsymbol{Q_1},\boldsymbol{Q_2}$.

Note If $\omega=1$, the RQTBC will reduce to the quadratic trigonometric Bézier curve with four shape parameters. If $\lambda=\mu=0$, the RQTBC will reduce to the rational quadratic trigonometric Bézier curve with two shape parameters in [15]. If $\alpha=\beta=-1,\lambda=\mu=0$, the RQTBC will reduce to $R(t)=\cos^2\Big(\frac{\pi}{2}t\Big)\boldsymbol{Q_0}+\sin^2\Big(\frac{\pi}{2}t\Big)\boldsymbol{Q_2}, t\in[0,1]$, which is a straight line between the control points $\boldsymbol{Q_0}$ and $\boldsymbol{Q_2}$.

4 Shape control of the RQTBC

The presence of shape parameters and the weight provide an intuitive control on the shape of the curve.

Figure 2 illustrates the effects of the shape parameters and the weight on the RQTBC. Figure 2 (a) shows the curves with four fixed $\beta=\lambda=0,\mu=\frac{1}{2},\omega=2$, $\alpha=-\frac{1}{2}$ (long dashed lines), 0 (solid lines) and 1 (short dashed lines). Figure 2 (b) shows the curves with four fixed $\alpha=\mu=0,\lambda=\frac{1}{2},\omega=2$, $\beta=-\frac{1}{2}$ (long dashed lines), 0 (solid lines) and 1 (short dashed lines). Figure 2 (c) shows the curves with four fixed $\alpha=-1,\beta=1,\mu=0,\omega=2$, $\lambda=0$ (solid lines), 2 (long dashed lines) and 4 (short dashed lines). Figure 2 (d) shows the curves with four fixed $\alpha=\frac{1}{2},\beta=-\frac{1}{2},\lambda=0,\omega=2$, $\mu=0$ (solid lines), 1 (long dashed lines) and 5 (short dashed lines). Figure 2 (e) shows the curves with four fixed $\alpha=0,\beta=1,\mu=\lambda=\frac{1}{2}$, $\omega=1$(long dashed lines), 2 (solid lines) and 4 (short dashed lines).

For the same shape parameters $\alpha=m$ and $\beta=n$, it is shown from Figure 2 (c) and (d) that the RQTBC is closer to the control polygon than the curve in [15] by altering the values of λ and μ of the exponential functions.

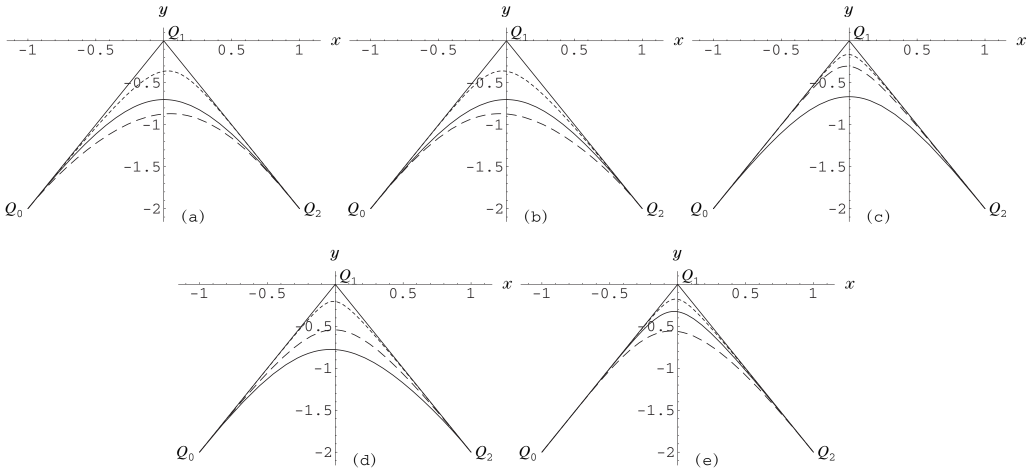

Figure 2 Effect on the shape of the curve with different parameters and weight

If $\lambda \to +\infty$, the RQTBC will reduce to a straight line between the control points $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$. If $\mu \to +\infty$, the RQTBC will reduce to a straight line between the control points $\boldsymbol{Q_0}$ and $\boldsymbol{Q_1}$.

Given the proper control points, the shape parameters and the weight, the corresponding RQTBC can be used to represent some special curves exactly, such as ellipses, circles, parabolas, and line segments. Figure 3 shows the arc of an ellipse, the arc of a circle and the arc of a parabola.

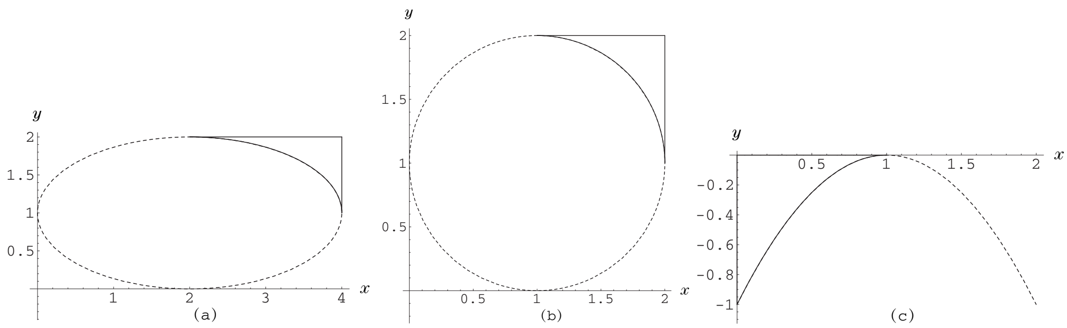

Figure 3 The representation of conics with RQTBC. (a) ellipse; (b) circle; (c) parabola

5 Continuity of the curves

Let a RQTBC $r(t)$ be given as (2) and a second RQTBC $r^*(t)$ be defined by

$$r^*(t) = \frac{b_0(t)\boldsymbol{V_0} + \omega^* b_1(t)\boldsymbol{V_1} + b_2(t)\boldsymbol{V_2}}{b_0(t) + \omega^* b_1(t) + b_2(t)}, \tag{3}$$

where the weight $\omega^* > 0$, four shape parameters $\alpha^*, \beta^* \in [-1, 1]$, $\lambda^*, \mu^* \in [0, +\infty)$.

Theorem 3 Given two segments of $r(t)$ and $r^*(t)$, then the necessary and sufficient condition of continuity is

(i) for C^0 continuity, $\boldsymbol{V_0} = \boldsymbol{Q_2}$;

(ii) for C^1 continuity, $\boldsymbol{V_1} = (1+m)\boldsymbol{Q_2} - m\boldsymbol{Q_1},\ m = \dfrac{\left(\frac{\pi}{2} + \frac{\pi}{2}\beta + \mu\right)\omega}{\left(\frac{\pi}{2} + \frac{\pi}{2}\alpha^* + \lambda^*\right)\omega^*}$;

(iii) for C^2 continuity, $\boldsymbol{V_2} = \boldsymbol{Q_0} + \nu(\boldsymbol{Q_2} - \boldsymbol{Q_1})$,

$$\nu = \frac{1}{k}(h\omega + l\omega^2 + m(h\omega^* + l\omega^{*2})),\ k = \frac{\pi^2}{4}\mathrm{e}^{-\lambda}(1-\alpha),\ l = 2\Big(\frac{\pi}{2} + \frac{\pi}{2}\beta + \mu\Big)^2$$
$$h = \frac{\pi^2}{4}\mathrm{e}^{-\lambda}(1-\alpha) - \Big(\frac{\pi^2}{2} + \frac{\pi^2}{2}\beta + \frac{\pi^2}{2}\beta^2 + \pi\mu + \pi\beta\mu + \mu^2\Big)$$

with $\alpha = \beta^*(\neq 1), \beta = \alpha^*, \lambda = \mu^*, \mu = \lambda^*$.

Proof The result (i) is obvious for $r^*(0) = r(1)$.

For C^1 continuity, the tangents of the two curves at the joint must be equal, that is,

$$\begin{cases} \boldsymbol{V_0} = \boldsymbol{Q_2}, \\ r^{*\prime}(0) = r'(1). \end{cases}$$

Then

$$\Big(\frac{\pi}{2} + \frac{\pi}{2}\alpha^* + \lambda^*\Big)\omega^*(\boldsymbol{V_1} - \boldsymbol{V_0}) = \Big(\frac{\pi}{2} + \frac{\pi}{2}\beta + \mu\Big)\omega(\boldsymbol{Q_2} - \boldsymbol{Q_1}).$$

The result (ii) holds after simple reorganization.

The two curves are joined by C^2 continuity if

$$\begin{cases} \boldsymbol{V_0} = \boldsymbol{Q_2}, \\ r^{*\prime}(0) = r'(1), \\ r^{*\prime\prime}(0) = r''(1). \end{cases}$$

Taking $\alpha = \beta^*(\neq 1), \beta = \alpha^*, \lambda = \mu^*, \mu = \lambda^*$, we obtain from the property (i) of Theorem 2,

$$k(\boldsymbol{V_2} - \boldsymbol{V_0}) + (h\omega^* + l\omega^{*2})(\boldsymbol{V_0} - \boldsymbol{V_1}) = k(\boldsymbol{Q_0} - \boldsymbol{Q_2}) + (h\omega + l\omega^2)(\boldsymbol{Q_2} - \boldsymbol{Q_1}).$$

Since $\boldsymbol{V_1} - \boldsymbol{V_0} = m(\boldsymbol{Q_2} - \boldsymbol{Q_1})$ and $\boldsymbol{V_0} = \boldsymbol{Q_2}$, the result (iii) follows.

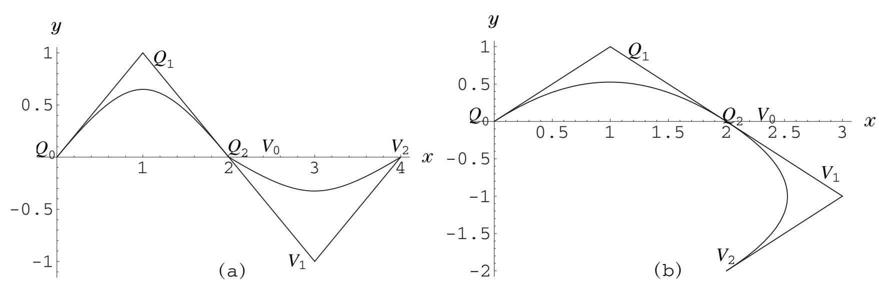

Figure 4 The continuity of the two curves (a) C^1 continuity (b) C^2 continuity

Figure 4 shows the continuity of the two curves. For C^1 continuous joint curve, the shape parameters are $\alpha = \beta = \lambda = \alpha^* = \beta^* = \mu^* = 0, \mu = \lambda^* = \frac{1}{2}, \omega = \omega^* = 2$. And for C^2 continuous joint curve, the shape parameters are $\alpha = \beta^* = \frac{1}{4}, \beta = \alpha^* = 0, \lambda = \mu^* = \frac{1}{2}, \mu = \lambda^* = 0, \omega = \omega^* = 1$.

6 Conclusion

The presented RQTBC has most of the similar geometric properties of the traditional rational quadratic Bézier curve and can accurately represent some conics. The composition of two curve segments using C^0, C^1 and C^2

continuity conditions is discussed. The shape of the curve can be adjusted by altering the values of shape parameters and the weight while the control polygon is kept unchanged. The new curves can be freely adopted in CAD/CAM systems.

References:

[1] Zhang J W. C-curves: An Extension of Cubic Curves [J]. Comput Aided Geom D, 1996, 13(3):199-217.

[2] Chen Q Y, Wang Z. A Class of Bézier-like Curves [J]. Comput Aided Geom D, 2003, 20(1):29-39.

[3] Yan L L, Liang J F. An Extension of the Bézier Model [J]. Appl Math Comput, 2011, 218(6):2863-2879.

[4] Han X L. Quadratic Trigonometric Polynomial Curves Concerning Local Control [J]. Appl Numer Math, 2006, 56(1):105-115.

[5] Han X L, Zhu Y P. Curve Construction Based on Five Trigonometric Blending Functions [J]. BIT Numer Math, 2012, 52(4):953-979.

[6] Han X L. Normalized B-basis of the Space of Trigonometric Polynomials and Curve Design [J]. Appl Math Comput, 2015, 251(C):336-348.

[7] Hoffmann M, Juhász I, Károlyi G. A Control Point Based Curve with Two Exponential Shape Parameters [J]. BIT Numer Math, 2014, 54(3):691-710.

[8] Zhu Y, Han X L. Curves and Surfaces Construction Based on New Basis with Exponential Functions [J]. Acta Appl Math, 2014, 129(1):183-203.

[9] Han X L. Quadratic Trigonometric Polynomial Curves with a Shape Parameter [J]. Comput Aided Geom D, 2002, 19(7):503-512.

[10] Han X L. Cubic Trigonometric Polynomial Curves with a Shape Parameter [J]. Comput Aided Geom D, 2004, 21(6):535-548.

[11] Han X A, Ma Y, Huang X. The Cubic Trigonometric Bézier Curve with Two Shape Parameters [J]. Appl Math Lett, 2009, 22(2):226-231.

[12] Sheng M, Su B. A Class of Bézier-type Curves and Surfaces by Trigonometric Polynomials [J]. J Comput Inform Syst, 2013, 9(1):89-96.

[13] Bashir U, Abbas M, Awang M, et al. A Class of Quasi-quintic Trigonometric Bézier Curve with Two Shape Parameters [J]. ScienceAsia, 2013, 39S(1):11-15.

[14] Hussain M, Saleem S. C^1 Rational Quadratic Trigonometric Spline [J]. Egypt Inform J, 2013, 14(3):211-220.

[15] Bashir U, Abbas M, Ali J. The G^2 and C^2 Rational Quadratic Trigonometric Bézier Curve with Two Shape Parameters with Applications [J]. Appl Math Comput, 2013, 219(20):10183-10197.

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