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A geometric feature of the Newton law of gravitation

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Abstract: In the Newton law of gravitation, the most miraculous fact is that the gravity is reciprocally proportional to the square of the distance between particles. In this paper, by assuming that the gravity is along with the line passing through particles and is proportional to the product of masses of particles, we will show that the above fact is equivalent to the geometric requirement that the gravity between two homogeneous balls is equal to that between two particles of the same masses located at the centers of balls. In fact, this will lead to a second-order Euler equation whose physical solution is reciprocally proportional to the square of the distance.

Key words: Newton law of gravitation; geometric feature; integral equation; Euler equation

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牛顿引力定律的一个几何特征

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摘　要：在牛顿的引力定律中，最为重要的事实是引力的大小是反比于质点之间的距离的平方. 基于引力是沿着质点的连线方向并正比于质量的乘积之前提下，证明了引力的反平方距离的事实完全等价于两个均匀球体之间的引力可以归结于位于球心处的同质量质点之间的引力. 事实上，这个几何要求将导致一个二阶的线性欧拉方程，其具有物理意义的解恰好是反比于距离的平方.

关键词：牛顿引力定律；几何特征；积分方程；欧拉方程

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1 The statement

Based on the three laws of Kepler on motions of planets in the solar system, Newton has proposed three laws on motions of particle systems. By considering spherical planets as particles located at the centers of balls, Newton has applied his invention of Calculus of Differentiation and Integration to give a rigorous mathematical deduction and then to obtain the gravitation law. This is the greatest achievement and the starting point of modern sciences. All of these have been given in Newton′s masterpiece Mathematical Principles of Natural Philosophy[1].

The gravitation law is now elementary to school boys and girls and can be stated as follows. Given two particles of masses m_i located at $p_i \in \mathbb{R}^3$, $i = 1, 2$, where $p_1 \neq p_2$, the gravity of the particle m_1 by the particle m_2 is such a vector $\vec{F} \in \mathbb{R}^3$ that

$(\mathbf{F}_1)$ $\vec{F}$ is along with the direction from p_1 to p_2,

$(\mathbf{F}_2)$ $\|\vec{F}\|$ is proportional to the product of the masses, and

$(\mathbf{F}_3)$ $\|\vec{F}\|$ is reciprocally proportional to the square of the distance between the particles.

Mathematically, by denoting

$$\vec{r} = p_2 - p_1 \in \mathbb{R}^3 \setminus \{O\}, \qquad r = \|\vec{r}\| > 0, \tag{1}$$

the gravity $\vec{F}$ is given by

$$\vec{F} = \frac{Gm_1m_2}{r^2}\frac{\vec{r}}{r}, \tag{2}$$

where $G > 0$ is the gravity constant.

With the gravitation law (2) for particles at hand, it is a standard fact that the gravity between two homogenous balls can be reduced to that for two particles of the masses same as the balls located at the centers of balls. Moreover, by solving the motion equations of particles, one can recover the Kepler′s three laws for motions of planets, as did in the usual textbooks on Ordinary Differential Equations, See [2].

The most miraculous conclusion in the gravitation law (2) is Fact $(\mathbf{F}_3)$, i.e., $\|\vec{F}\|$ is reciprocally proportional to r^2, the square of the distance between the particles. In this paper, based on Facts $(\mathbf{F}_1)$ and $(\mathbf{F}_2)$, we will show that Fact $(\mathbf{F}_3)$ is equivalent to the following geometric requirement.

$(\mathbf{F}_4)$ The gravity between two homogeneous balls is equal to that between two particles of the same masses located at the centers of the balls.

Let us describe this in a mathematical way. Given two particles of masses m_i located at $p_i \in \mathbb{R}^3$, $i = 1, 2$, $p_1 \neq p_2$, from Facts $(\mathbf{F}_1)$ and $(\mathbf{F}_2)$, the gravity $\vec{F}$ between them is given by

$$\vec{F} = m_1m_2f(r)\frac{\vec{r}}{r}. \tag{3}$$

Here $\vec{r}$ and r are as in (1), and $f : (0, \infty) \to (0, \infty)$ is a function to be determined. From the physical explanation to (3), it is reasonable to assume that

$(\mathbf{A})$ The function $f(r)$ is continuous and strictly decreasing.

By using formula (3), for two homogenous matters of total masses m_i located in the spatial domains $\Omega_i \subset \mathbb{R}^3$, $i = 1, 2$, $\Omega_1 \cap \Omega_2 = \emptyset$, the gravity between them is given by the following multi-dimensional integral

$$\vec{F} = \frac{m_1m_2}{\mu(\Omega_1)\mu(\Omega_2)}\int_{\Omega_1\times\Omega_2} f(\|p_2 - p_1\|)\frac{p_2 - p_1}{\|p_2 - p_1\|}\mathrm{d}p_1\mathrm{d}p_2. \tag{4}$$

Here $\mu(\Omega_i)'$s are volumes of Ω_i. In particular, if

$$\Omega_i = B_{R_i}(p_i^0) := \left\{p \in \mathbb{R}^3 : \|p - p_i^0\| \leqslant R_i\right\}, \qquad i = 1, 2,$$

are balls, where

$$\|p_2^0 - p_1^0\| > R_1 + R_2, \tag{5}$$

then (4) is

$$\vec{F} = \frac{m_1 m_2}{\dfrac{4\pi R_1^3}{3}\dfrac{4\pi R_2^3}{3}} \int_{B_{R_1}(p_1^0)\times B_{R_2}(p_2^0)} f(\|p_2 - p_1\|)\frac{p_2 - p_1}{\|p_2 - p_1\|}\mathrm{d}p_1\mathrm{d}p_2.$$

Thus the geometric requirement $(\mathbf{F}_4)$ is the following equation for the function f

$$\frac{m_1 m_2}{\dfrac{4\pi R_1^3}{3}\dfrac{4\pi R_2^3}{3}} \int_{B_{R_1}(p_1^0)\times B_{R_2}(p_2^0)} f(\|p_2 - p_1\|)\frac{p_2 - p_1}{\|p_2 - p_1\|}\mathrm{d}p_1\mathrm{d}p_2 m_1 m_2 f(\|p_2^0 - p_1^0\|)\frac{p_2^0 - p_1^0}{\|p_2^0 - p_1^0\|} \tag{6}$$

for all parameters as in (5). The main result is as follows.

Theorem Under assumption $(\mathbf{A})$ on f, the unique solution of Eq. (6) is given by

$$f(r) = \frac{k}{r^2} \qquad \forall r \in (0, \infty), \tag{7}$$

where k is a positive constant.

2 The proof

Eq. (6) is a linear integral equation for f, where the masses m_i are irrelevant. Henceforth we assume that $m_1 = m_2 = 1$.

Now we give a rigorous reduction to Eq. (6). First of all, the geometric requirement $(\mathbf{F}_4)$ can be reduced to the case that one of the balls shrinks into a particle, say that $B_{R_2}(p_2^0)$ is a point p_2^0. Going back to Eq. (6), this can be stated as follows.

Lemma The function f satisfies Eq. (6) for all parameters as in (5) if and only if f satisfies

$$\frac{1}{\dfrac{4\pi R_1^3}{3}} \int_{B_{R_1}(p_1^0)} f(\|p_2^0 - p_1\|)\frac{p_2^0 - p_1}{\|p_2^0 - p_1\|}\mathrm{d}p_1 = f(\|p_2^0 - p_1^0\|)\frac{p_2^0 - p_1^0}{\|p_2^0 - p_1^0\|} \tag{8}$$

for all parameters satisfying

$$\|p_2^0 - p_1^0\| > R_1. \tag{9}$$

Proof The sufficiency. Let us assume (8) for all balls and all particles. To obtain (6), by considering $\mathrm{d}p_2$ as a particle located at $p_2 \in B_{R_2}(p_2^0)$ of the mass

$$\frac{m_2}{\dfrac{4\pi R_2^3}{3}}\mathrm{d}p_2,$$

it follows from assumption (8) that

$$\frac{m_1 m_2}{\dfrac{4\pi R_1^3}{3}\dfrac{4\pi R_2^3}{3}} \int_{B_{R_1}(p_1^0)} f(\|p_2 - p_1\|)\frac{p_2 - p_1}{\|p_2 - p_1\|}\mathrm{d}p_1\mathrm{d}p_2 = \frac{m_1 m_2}{\dfrac{4\pi R_2^3}{3}} f(\|p_2 - p_1^0\|)\frac{p_2 - p_1^0}{\|p_2 - p_1^0\|}\mathrm{d}p_2.$$

With this at hand, by applying (8) again, one has

$$\begin{aligned}&\frac{m_1m_2}{\frac{4\pi R_1^3}{3}\frac{4\pi R_2^3}{3}}\int_{B_{R_1}(p_1^0)\times B_{R_2}(p_2^0)}f(\|p_2-p_1\|)\frac{p_2-p_1}{\|p_2-p_1\|}\mathrm{d}p_1\mathrm{d}p_2\\ =&\int_{B_{R_2}(p_2^0)}\frac{m_1m_2}{\frac{4\pi R_2^3}{3}}f(\|p_2-p_1^0\|)\frac{p_2-p_1^0}{\|p_2-p_1^0\|}\mathrm{d}p_2\\ =&\,m_1m_2f(\|p_2^0-p_1^0\|)\frac{p_2^0-p_1^0}{\|p_2^0-p_1^0\|}.\end{aligned}$$

Thus Eq. (6) is obtained.

The necessity. Let us take $m_1=m_2=1$ in (6). By taking the limit of (6) as $R_2\to 0$, one can obtain Eq. (8).

Now we give the proof of the theorem. Due to the lemma above, we need only to solve Eq. (8). As for parameters in (9), without loss of generality, let us take $p_1^0=O$, the origin, and $p_2^0=(0,0,L)$ in the upper z-axis. By rewriting R_1 as R, we know that L and R satisfy

$$L>R>0. \tag{10}$$

Then the left-hand side of (8) is

$$\vec{F}=\frac{1}{\frac{4\pi R^3}{3}}\iiint_{B_R(O)}f\left(\sqrt{x^2+y^2+(z-L)^2}\right)\frac{(-x,-y,L-z)}{\sqrt{x^2+y^2+(z-L)^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Due to the symmetry of $B_R(O)$, one has

$$\vec{F}=(0,0,F_z),$$

where

$$F_z=\frac{1}{\frac{4\pi R^3}{3}}\iiint_{x^2+y^2+z^2\leqslant R^2}f\left(\sqrt{x^2+y^2+(z-L)^2}\right)\frac{L-z}{\sqrt{x^2+y^2+(z-L)^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{11}$$

The right-hand side of (8) is simply $(0,0,f(L))$. Hence Eq. (8) is reduced to

$$F_z=f(L). \tag{12}$$

The quantity F_z of (11) can be reduced to

$$\begin{aligned}F_z=&\frac{1}{\frac{4\pi R^3}{3}}\int_{-R}^{R}\left(\iint_{x^2+y^2\leqslant R^2-z^2}f\left(\sqrt{x^2+y^2+(z-L)^2}\right)\frac{\mathrm{d}x\mathrm{d}y}{\sqrt{x^2+y^2+(z-L)^2}}\right)(L-z)\mathrm{d}z\\ =&\frac{1}{\frac{4\pi R^3}{3}}\int_{-R}^{R}\left(\int_0^{\sqrt{R^2-z^2}}\int_0^{2\pi}f\left(\sqrt{\rho^2+(z-L)^2}\right)\frac{\rho\mathrm{d}\rho\mathrm{d}\theta}{\sqrt{\rho^2+(z-L)^2}}\right)(L-z)\mathrm{d}z\\ &\qquad(\text{by setting } x=\rho\cos\theta,\ y=\rho\sin\theta)\\ =&\frac{1}{\frac{2R^3}{3}}\int_{-R}^{R}\left(\int_0^{\sqrt{R^2-z^2}}f\left(\sqrt{\rho^2+(z-L)^2}\right)\frac{\rho\mathrm{d}\rho}{\sqrt{\rho^2+(z-L)^2}}\right)(L-z)\mathrm{d}z\\ =&\frac{1}{\frac{2R^3}{3}}\int_{-R}^{R}\left(F\left(\sqrt{\rho^2+(z-L)^2}\right)\Big|_{\rho=0}^{\rho=\sqrt{R^2-z^2}}\right)(L-z)\mathrm{d}z\end{aligned}$$

$$= \frac{1}{\frac{2R^3}{3}} \int_{-R}^{R} \left((L-z)F\left(\sqrt{L^2+R^2-2Lz}\right) - (L-z)F(L-z) \right) \mathrm{d}z, \tag{13}$$

where $F : (0, \infty) \to \mathbb{R}$ is an anti-derivative of f

$$F'(s) = f(s) \qquad \forall s \in (0, \infty). \tag{14}$$

By using the change $s = \sqrt{L^2 + R^2 - 2Lz}$ for variables, one has

$$\int_{-R}^{R} (L-z)F\left(\sqrt{L^2+R^2-2Lz}\right) \mathrm{d}z = \int_{L-R}^{L+R} \frac{s^3 + (L^2-R^2)s}{2L^2} F(s)\mathrm{d}s,$$

and by using the change $s = L - z$ for variables, one has

$$\int_{-R}^{R} (L-z)F(L-z)\,\mathrm{d}z = \int_{L-R}^{L+R} sF(s)\mathrm{d}s.$$

Substituting into (13), we obtain

$$F_z = \frac{1}{\frac{4R^3L^2}{3}} \left(\int_{L-R}^{L+R} s^3F(s)\mathrm{d}s - (L^2+R^2) \int_{L-R}^{L+R} sF(s)\mathrm{d}s \right).$$

As a consequence, Eq. (12) is

$$\int_{L-R}^{L+R} s^3F(s)\mathrm{d}s - (L^2+R^2) \int_{L-R}^{L+R} sF(s)\mathrm{d}s = \frac{4R^3}{3} L^2 f(L), \tag{15}$$

where L and R, as in (10), are arbitrary.

Some observation on Eq. (15) is as follows. In case $f(s) \equiv 1/s^2$, by taking $F(s) = -1/s$, the left-hand side of (15) is

$$\int_{L-R}^{L+R} -s^2\mathrm{d}s - (L^2+R^2) \int_{L-R}^{L+R} -\mathrm{d}s = \frac{4R^3}{3} = \frac{4R^3}{3} L^2 f(L).$$

Hence $f(s) = 1/s^2$ satisfies Eq. (15). In case $f(s) \equiv s$, by taking $F(s) = s^2/2$, the left-hand side of (15) is

$$\int_{L-R}^{L+R} \frac{s^5}{2}\mathrm{d}s - (L^2+R^2) \int_{L-R}^{L+R} \frac{s^3}{2}\mathrm{d}s = \frac{4R^3L^3}{3} = \frac{4R^3}{3} L^2 f(L).$$

We know that $f(s) = s$ also satisfies Eq. (15).

Due to assumption $(\mathbf{A})$, f is continuous. By (14), F is of class C^1. Thus the left-hand side of (15) is of class C^2 in L. Hence Eq. (15) shows that f is of class C^2. Inductively, we conclude from Eq. (15) that f is of class C^∞.

Differentiating (15) with respect to R, it follows from the Newton-Leibnitz formula that

$$(L+R)^3F(L+R) + (L-R)^3F(L-R) - 2R \int_{L-R}^{L+R} sF(s)\mathrm{d}s$$

$$-(L^2+R^2)\left((L+R)F(L+R) + (L-R)F(L-R)\right) = 4R^2L^2f(L),$$

i.e.,

$$2RL^2f(L) = L(L+R)F(L+R) - L(L-R)F(L-R) - \int_{L-R}^{L+R} sF(s)\mathrm{d}s.$$

Differentiating it with respect to R again and using (14), we have

$$\begin{aligned}2L^2 f(L) &= LF(L+R) + LF(L-R) + L(L+R)f(L+R) + L(L-R)f(L-R) \\ &\quad -((L+R)F(L+R) + (L-R)F(L-R)) \\ &= -RF(L+R) + RF(L-R) + L(L+R)f(L+R) + L(L-R)f(L-R).\end{aligned}$$

Further differentiation with respect to R gives

$$\begin{aligned}0 &= -F(L+R) + F(L-R) + (L-R)f(L+R) - (L+R)f(L-R) \\ &\quad +L(L+R)f'(L+R) - L(L-R)f'(L-R).\end{aligned}$$

Differentiating it with respect to R again, we have

$$\begin{aligned}0 &= L(L+R)f''(L+R) + L(L-R)f''(L-R) \\ &\quad +(2L-R)f'(L+R) + (2L+R)f'(L-R) - 2f(L+R) - 2f(L-R).\end{aligned}$$

By letting $R \downarrow 0$ in this equation, we obtain

$$0 = 2L^2 f''(L) + 4Lf'(L) - 4f(L).$$

By denoting $r := L$, this is a second-order Euler equation

$$r^2 \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}f}{\mathrm{d}r} - 2f = 0. \tag{16}$$

Eq. (16) can be solved by the standard change $s = \log r \in \mathbb{R}$ for variables, which leads to

$$\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + \frac{\mathrm{d}f}{\mathrm{d}s} - 2f = 0.$$

Solving this equation, we obtain

$$f = ke^{-2s} + \hat{k}e^{s}.$$

Hence solutions of Eq. (16) are

$$f(r) = \frac{k}{r^2} + \hat{k}r, \tag{17}$$

where k and $\hat{k}$ are constants. From the observation above, all functions from (17) do satisfy Eq. (12), or Eq. (15).

Finally, as we are considering decreasing solutions $f(s)$, cf. assumption $(\mathbf{A})$, it is easy to exclude the case $\hat{k} \neq 0$ in (17). Hence we have the desired result (7) and complete the proof of the theorem.

Remark Eq. (16) can also be obtained by using the Taylor expansion. Let us rewrite Eq. (15) as

$$\begin{aligned}\frac{4R^3}{3}L^2 f(L) &= \int_{L-R}^{L+R} \left(s^3 - L^2 s\right) F(s)\mathrm{d}s - R^2 \int_{L-R}^{L+R} sF(s)\mathrm{d}s \\ &= \int_{-R}^{R} \left(2L^2 s + 3Ls^2 + s^3\right) F(L+s)\mathrm{d}s - R^2 \int_{-R}^{R} (L+s)\, F(L+s)\mathrm{d}s \\ &=: I_1 - I_2.\end{aligned} \tag{18}$$

As a function of $R \in (0, L)$, the left-hand side of (18) is cubic in R. Let us expand the right-hand side of (18) at $R = 0$. When $s \to 0$, one has

$$\begin{aligned}
&\left(2L^2 s + 3Ls^2 + s^3\right) F(L+s) \\
&= \left(2L^2 s + 3Ls^2 + s^3\right) \left(F(L) + F'(L)s + \frac{F''(L)}{2}s^2 + \frac{F'''(L)}{6}s^3 + o(s^3)\right) \\
&= 2L^2 F(L)s + (2L^2 F'(L) + 3LF(L))s^2 + (L^2 F''(L) + 3LF'(L) + F(L))s^3 \\
&\quad + \left(\frac{L^2 F'''(L)}{3} + \frac{3LF''(L)}{2} + F'(L)\right) s^4 + o(s^4).
\end{aligned}$$

Hence, as $R \downarrow 0$,

$$I_1 = \left(\frac{4L^2 F'(L)}{3} + 2LF(L)\right) R^3 + \left(\frac{2L^2 F'''(L)}{15} + \frac{3LF''(L)}{5} + \frac{2F'(L)}{5}\right) R^5 + o(R^5).$$

Similarly, one has

$$\begin{aligned}
(L+s)\, F(L+s) &= (L+s) \left(F(L) + F'(L)s + \frac{F''(L)}{2}s^2 + o(s^2)\right) \\
&= LF(L) + (LF'(L) + F(L))s + \left(\frac{LF''(L)}{2} + F'(L)\right) s^2 + o(s^2),
\end{aligned}$$

and

$$I_2 = 2LF(L)R^3 + \left(\frac{LF''(L)}{3} + \frac{2F'(L)}{3}\right) R^5 + o(R^5).$$

By recalling (14), we obtain

$$I_1 - I_2 = \frac{4}{3}L^2 f(L) R^3 + \frac{2}{15}\left(L^2 f''(L) + 2Lf'(L) - 2f(L)\right) R^5 + o(R^5).$$

Comparing with Eq. (18), we also arrive at the Euler equation (16) for f.

In conclusion, under $(\mathbf{F}_1)$ and $(\mathbf{F}_2)$, the gravity is reciprocally proportional to the square of the distance is the same as the geometric requirement that the gravity between two balls is equal to that between two particles of the same masses located at the centers of balls. Such a geometric requirement was presumably assumed by Newton because the planets have been considered as particles.

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