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The existence of n-order algebraic curve solutions of planar quadratic polynomial systems

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Abstract: In this short article, we show that for any given positive integer $n \geqslant 2$, there is a planar quadratic differential system having n-degree and $2n-$degree classical algebraic curve solutions.

Key words: quadratic differential system; algebraic curve solution; periodic solution; integrable system

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平面二次多项式系统中 n 阶代数曲线解的存在性

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摘　要: 用具体例子证明对于任给的正整数 $n \geqslant 2$, 平面二次多项式系统中，存在 n 阶和 $2n$ 阶代数曲线解.

关键词: 二次系统；代数曲线解；周期解；可积系统

1 Introduction

We are interested in the study of planar polynomial systems, because they occur very often in applications. Indeed, such equations appear in modelling chemical reactions, population dynamics, travelling wave systems of nonlinear evolution equations in mathematical physics and in many other areas of applied mathematics and mechanics. From the mathematical point of view, quadratic systems are perhaps the most simple nonlinear differential systems. Despite their simplicity, there are important open questions around them. It is a particular case of the famous Hilbert′s 16th problem[1].

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More recently, Garcia and Llibre[2] gave a lot of examples to show that classical planar algebraic curves are realizable by quadratic polynomial differential systems. We say that a given algebraic curve is realized by a quadratic system when this algebraic curve is an invariant curve solution of a system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=P_2(x,y)=\Sigma_{i+j=0}^{2}a_{ij}x^iy^j,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=Q_2(x,y)=\Sigma_{i+j=0}^{2}b_{ij}x^iy^j,\tag{1}$$

where a_{ij} and b_{ij} are constant parameters.

It is well known that any cubic algebraic curve is always realized by some quadratic Hamiltonian system[3−4]. Quadratic systems realizing classical quadratic and higher degree algebraic curves have been found.

For a given positive integer n, is there an n-degree algebraic curve such that it is realizable by quadratic polynomial differential systems? In this paper, we show the following conclusion.

Theorem 1 For any given positive integer $n \geqslant 2$, there is a planar quadratic differential system having n-degree and $2n-$degree classical algebraic curve solutions.

The proof of this theorem is given in section 2.

2 An integrable quadratic system having n-degree classical algebraic curve solution

Consider the quadratic differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-xy,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=ax^2+by^2-\frac{1}{4a}.\tag{2}$$

This is an integrable system with the first integral

$$H(x,y)=x^{2b}\left(y^2+\frac{a}{b+1}x^2-\frac{1}{4ab}\right)=h,\quad \text{for }\ b\neq -1;\tag{3}$$

$$H(x,y)=\frac{y^2}{x^2}+\frac{1}{4ax^2}+2a\ln(x),\quad \text{for }\quad b=-1.\tag{4}$$

We only consider the case $a>0$. Otherwise, we apply the transformation $x\to -x, y\to -y$.

Clearly, system (2) always has two singular points $E_{\mp}\Big(\mp\dfrac{1}{2a},0\Big)$. When $ab>0$, there exist two singular points $S_{\mp}\Big(0,\mp\dfrac{1}{2\sqrt{ab}}\Big)$ in the straight line $x=0$.

Let $M(x_j,y)$ be the coefficient matrix of the linearized system for equation (2) at a singular point. We have

$$J\left(\mp\frac{1}{2a},0\right)=\det M\left(\mp\frac{1}{2a},0\right)=\frac{1}{2a}>0,\ J\left(0,\mp\frac{1}{2\sqrt{ab}}\right)=\det M\left(0,\mp\frac{1}{2\sqrt{ab}}\right)=-\frac{1}{2a}<0.$$

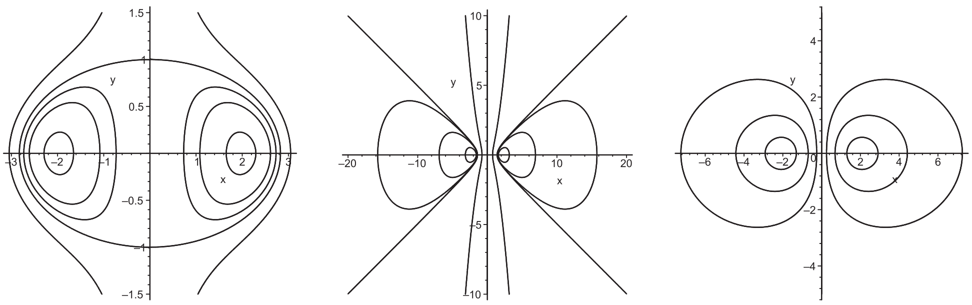
When $b\neq -1$, we write that

$$h_1=H\left(-\frac{1}{2a},0\right)=-\frac{1}{4a}\left(-\frac{1}{2a}\right)^{2b}\left(\frac{1}{b}-\frac{1}{b+1}\right),$$

$$h_2=H\left(\frac{1}{2a},0\right)=-\frac{1}{4a}\left(\frac{1}{2a}\right)^{2b}\left(\frac{1}{b}-\frac{1}{b+1}\right),\ h_s=H\left(0,\frac{1}{2\sqrt{ab}}\right)=0.$$

By using the above information, we have the following qualitative analysis.

Lemma 1 The planar quadratic system (2) has two symmetric centers with respect to the $y-$coordinate axis, for which there exist exactly three topological phase portraits shown in Figs. 1(a), 1(b) and 1(c).



(a) $b > 0$ (b) $b < -1$ (c) $-1 < b < 0$

Figure 1 The phase portraits of the system (2) for $a > 0$

Lemma 2 Take $a > 0, b = \frac{1}{2}(n-2) > 0$, where n is a positive integer. Then the n-degree algebraic curves defined by

$$H_+(x,y) = x^{n-2}\left(y^2 + \frac{a}{b+1}x^2 - \frac{1}{4ab}\right) - h = 0, \ \ h \in (h_2, 0) \tag{5}$$

give rise to two periodic solutions of system (2), see Fig. 1(a).

Lemma 3 Take $a > 0, b = -n < -1$, where n is a positive integer. Then the $2n-$degree algebraic curves defined by

$$H_-(x,y) = \left(y^2 + \frac{a}{b+1}x^2 - \frac{1}{4ab}\right) - hx^{2n} = 0, \ \ h \in (h_2, 0)$$

give rise to two periodic solutions of system (2), see Fig. 1(b).

Thus we have proved Theorem 1 by Lemma 2 and Lemma 3.

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