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Linearizability conditions for $1:-5$ Lotka-Volterra two-dimensional complex quartic systems

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Abstract: In this paper we investigate the linearizability problem for the planar Lotka–Volterra complex quartic systems which are $1:-5$ linear systems perturbed by homogeneous polynomials of degree 4, that is to say, we consider systems of the form $\dot{x} = x(1 - a_{30}x^3 - a_{21}x^2y - a_{12}xy^2 - a_{03}y^3)$, $\dot{y} = -y(5 - b_{30}x^3 - b_{21}x^2y - b_{12}xy^2 - b_{03}y^3)$. The necessary and sufficient conditions for the linearizability of this system are found.

Key words: polynomial systems; linearization; invariant curve; cofactor

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二维 $1:-5$ Lotka-Volterra 复四次系统的可线性化条件

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摘 要：本文考虑了如下的一类平面四次复 Lotka-Volterra 系统的可线性化问题

$$\dot{x} = x(1 - a_{30}x^3 - a_{21}x^2y - a_{12}xy^2 - a_{03}y^3),\ \dot{y} = -y(5 - b_{30}x^3 - b_{21}x^2y - b_{12}xy^2 - b_{03}y^3).$$

该系统为四次齐次多项式扰动下的具有 $1:-5$ 线性项的复 Lotka-Volterra 系统，给出了该系统可线性化的充分必要条件.

关键词：多项式系统；可线性化；不变曲线；余因式

1 Introduction

Consider a real planar analytic differential system with an isolated singular point at the origin, at which the eigenvalues of the linear part are non-zero pure imaginary numbers. By analytic change of coordinates and a constant

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time rescaling the system takes the form

$$\dot{u} = -v + \cdots, \quad \dot{v} = u + \cdots. \tag{1}$$

By the Poincaré–Lyapunov theorem, system (1) has a center at the origin if and only if there is a first integral $\Phi(u,v) = u^2 + v^2 + \cdots$ (see for instance [1] and the references therein).

If the origin is a center, then the problem that arises is to determine when the period of the solutions near the origin is constant. A center with such a property is called an isochronous center. By the isochronous center theorem of Poincaré and Lyapunov, the center of (1) is isochronous if and only if it is linearizable. Hence, the isochronicity problem is equivalent to the linearizability problem. Furthermore, it can be generalized to the case of complex systems as follows. Introducing a complex structure on phase plane $(u,\, v)$ by setting $x = u + iv, y = \bar{x} = u - iv$, after a change of time $idt = d\tau$ and rewriting t instead of τ, system (1) becomes a system of two complex differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \sum_{i+j=1}^{n-1} a_{i,j} x^{i+1} y^j, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -y + \sum_{i+j=1}^{n-1} b_{i,j} x^i y^{j+1}. \tag{2}$$

The origin is the $1:-1$ resonant singular point of system (2).

The next natural generalization of the result above is to consider the case of an analytic vector field on $\mathbb{C}^2$ with a $p:-q$ resonant elementary singular point at the origin, that is, a system of the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= px - \sum_{i+j=1}^{n-1} a_{i,j} x^{i+1} y^j = P(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -qy + \sum_{i+j=1}^{n-1} b_{i,j} x^i y^{j+1} = Q(x,y), \end{aligned} \tag{3}$$

where $p,\, q \in \mathbb{N}$ and $GCD(p,q) = 1$. We denote the coefficients by

$$(A,B) = (a_{1,0}, a_{0,1}, \cdots, a_{-1,n}, b_{0,1}, b_{1,0}, \cdots, b_{n,-1}).$$

We know that

Definition 1 [2] (i) The system (3) is integrable at the origin if there exists an analytic change of coordinates

$$(X,Y) = (x + o(x,y), y + o(x,y)) \tag{4}$$

bringing the system (3) to the system

$$\dot{X} = pXh(X,Y), \ \dot{Y} = -qYh(X,Y),$$

where $h(X,Y) = 1 + O(X,Y)$ ($X^qY^p = x^qy^p + h.o.t.$ is then a first integral of the type introduced by Dulac). The definition is also valid for the case $pq = 0$.

Moreover, the system (3) is integrable at the origin if it admits a first integral of the form

$$\Psi(x,y) = x^q y^p + \sum_{i+j=p+q+1}^{\infty} v_{i,j}(A,B) x^i y^j. \tag{5}$$

(ii) The system (3) is linearizable at the origin if there exists an analytic change of coordinates

$$z_1 = x + \sum_{i+j=2}^{\infty} c_{i-1,j}(A,B) x^i y^j, \ z_2 = y + \sum_{i+j=2}^{\infty} d_{i,j-1}(A,B) x^i y^j \tag{6}$$

bringing the system (3) to the system

$$\dot{z}_1 = pz_1, \ \dot{z}_2 = -qz_2. \tag{7}$$

The linearizability problem for the complex system (2) is a generalization of the linearizability (isochronicity) problem for the real system (1). Several methods have been developed to compute the necessary conditions to have an isochronous center, see [3–6] and references therein. However, There are only a few families of polynomial differential systems in which a complete classification of the linearizability (isochronicity) is known, see for instance [7–9].

Therefore, more works focused on the integrability and linearizability for the following planar polynomial systems,

$$\frac{\mathrm{d}x}{\mathrm{d}t}=x(p-P(x,y)),\quad \frac{\mathrm{d}y}{\mathrm{d}t}=-y(q-Q(x,y)). \tag{8}$$

System (8) has two invariant lines passing through the origin, and we call it a Lotka-Volterra system. The differential equations modeling the interaction of the two species have been studied extensively by real systems of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t}=xF(x,y),\quad \frac{\mathrm{d}y}{\mathrm{d}t}=yG(x,y),$$

also known as Kolmogonov systems.

Even in the case of quadratic nonlinearities, it is already necessary to restrict the class of systems under consideration. For example, for a $1:-q$ resonant quadratic system, indeed, complete results about integrability are known only for Lotka–Volterra systems:

Theorem 1 [2] For $q\in\mathbb{N}/\{1\}$ the Lotka$-$Volterra system $\dot{x}=x(1+A_1x+B_1y),\ \dot{y}=y(-q+A_2x+B_2y)$ has a resonant center at the origin if and only if

$$A_2[A_1+A_2][2A_1+A_2]\cdots[(q-2)A_1+A_2][qA_1B_1-(q-1)A_1B_2-A_2B_2]=0.$$

Recently, the integrability and linearizability problem for some Lotka-Volterra system having homogeneous nonlinearities with higher degree are considered. The $1:-q$ cubic Lotka–Volterra system was studied in [10–11]; linearizability of the $1:-1$ quartic Lotka–Volterra system was considered in [12]; integrability of the $1:-1$ quintic Lotka–Volterra system was considered in [13–14].

In this paper we study the linearizability problem for a class of quartic systems with $1:-5$ resonant saddle, namely the corresponding completely quartic homogeneous Kolmogonov system,

$$\begin{aligned}\dot{x}&=x(1-a_{30}x^3-a_{21}x^2y-a_{12}xy^2-a_{03}y^3),\\ \dot{y}&=-y(5-b_{30}x^3-b_{21}x^2y-b_{12}xy^2-b_{03}y^3).\end{aligned}\tag{9}$$

In the work we obtain necessary and sufficient conditions for (9) to be linearizable at the origin.

2 Preliminaries

In this section, we briefly describe the general approach to study the integrability and linearizability problem for the polynomial system (3). The first step is the calculation of the so-called linearizability quantities, which are polynomials of the coefficients $a_{i,j}$ and $b_{i,j}$ of system (3). Taking derivatives with respect to t in both parts of each of the equalities in (7) and equating coefficients of the terms $x^{q_1+1}y^{q_2}$ for the first equation and $x^{q_1}y^{q_2+1}$ for the second equation we obtain the recurrence formulae

$$(pq_1-qq_2)c_{q_1,q_2}=\sum_{s_1+s_2=0}^{q_1+q_2-1}[(s_1+1)c_{s_1,s_2}a_{q_1-s_1,q_2-s_2}-s_2c_{s_1,s_2}b_{q_1-s_1,q_2-s_2}], \tag{10}$$

$$(pq_1-qq_2)d_{q_1,q_2}=\sum_{s_1+s_2=0}^{q_1+q_2-1}[s_1d_{s_1,s_2}a_{q_1-s_1,q_2-s_2}-(s_2+1)d_{s_1,s_2}b_{q_1-s_1,q_2-s_2}], \tag{11}$$

where $s_1, s_2 \geqslant -1$, $q_1, q_2 \geqslant -1$, $q_1 + q_2 \geqslant 1$, $c_{1,-1} = c_{-1,1} = d_{1,-1} = d_{-1,1} = 0$, $c_{0,0} = d_{0,0} = 1$ and we set $a_{p,q} = a_{q,p} = 0$ if $p + q < 1$. Hence, we compute the c_{q_1,q_2} and d_{q_1,q_2} of the formal change of variables (6) step by step using the fomulae (10) and (11). In the case $pq_1 = qq_2 = kpq$ the coefficients $c_{qk,pk}$ and $d_{qk,pk}$ can be chosen arbitrarily (we set $c_{qk,pk} = d_{qk,pk} = 0$). The system is linearizable if and only if the quantities on the right-hand side of (10) and (11) are equal to zero for all $pq_1 = qq_2 = kpq$, $k \in \mathbb{N}$. In case $pq_1 = qq_2 = kpq$ we denote the polynomials on the right-hand side of (10) by i_k and on the right-hand side of (11) by $-j_k$, calling them the $k-$th linearizability quantities. Hence, system (3) with the given coefficient (A, B) is linearizable if, and only if, $i_k(A, B) = j_k(A, B) = 0$ for all $k \in \mathbb{N}$.

In the space of the parameters of a given family of systems (3) the set of all linearizable systems is an affine variety V of the ideal $< i_1, j_1, i_2, j_2, \cdots >$. We recall that the variety of a given set of polynomials $F =< f_1, f_2, \cdots, f_s >$ is the set of common zeros of the polynomials $f_1, f_2, \cdots, f_s$ and it is denoted by $V(F)$. Denote by I_k the ideal generated by the first k pairs of the linearizability quantities,

$$I_k =< i_1, j_1, i_2, j_2, \cdots, i_k, j_k > . \tag{12}$$

By the Hilbert basis theorem there exists $N \in \mathbb{N}$ such that $V = V(< i_1, j_1, i_2, j_2, \cdots >)$ is equal to the variety of the ideal I_N, $V = V(I_N)$. However, the theorem does not give a constructive procedure to find N. In practice, N is taken such that

$$V(I_N) = V(I_{N+1}). \tag{13}$$

Subsequently, the minimal associated primes of the ideal are computed. The computational tool we used is the routine minAssGTZ of the computer algebra system SINGULAR which computes the decomposition using the method described in [1]. For each component one tries to find the linearizing transformation (6).

The most powerful method to find linearizing transformations is the so-called Darboux linearization, see [1–2]. A smooth function $f(x, y)$ satisfying

$$\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} = kf \tag{14}$$

is called a Darboux factor of system (3) and the polynomial $k(x, y)$ is called the cofactor.

To construct a Darboux first integral for system (3), we have the following theorem.

Theorem 2 [1] If there are Darboux functions $f_1,\ f_2, \cdots, f_m$ with the cofactors $k_1,\ k_2, \cdots, k_m$ satisfying

$$\sum_{i=1}^{m} \alpha_i k_i = 0,$$

then $H = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}$ is a Darboux first integral of (3), and if

$$\sum_{i=1}^{m} \beta_i k_i + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0,$$

then the system admits the integrating factor $\mu = f_1^{\beta_1} f_2^{\beta_2} \cdots f_m^{\beta_m}$.

Moreover, if system (3) admits an integrating factor, then the system is integrable and admits a first integral of the form (5).

Obviously, a linearizable system (3) should be integrable. The following theorem allows us to construct linearizing substitutions if sufficiently many Darboux factors are known, and it also provides a method to find linearizing substitutions for integrable system.

Theorem 3 [2] The system (3) is linearizable if one of the following situations occurs:

Case I: There exist analytic functions $F_1(x, y), \cdots, F_m(x, y), K_1(x, y), \cdots, K_m(x, y)$ defined in a neighborhood of the origin and numbers $\alpha_1, \cdots, \alpha_{m-1}, \beta_2, \cdots, \beta_m$ satisfying $\dfrac{\partial F_i}{\partial x}\dot{x} + \dfrac{\partial F_i}{\partial y}\dot{y} = K_i F_i$ and

(i) $F_1(x,y) = x + o(x,y),\ F_2(x,y) = y + o(x,y),\ F_i(0,0) = 1$ for $i = 2, \cdots, m-1$.

(ii) $\sum\limits_{i=1}^{m-1} \alpha_i K_i = p$.

(iii) $\sum\limits_{i=2}^{m} \beta_i K_i = -q$.

The linearizing change of coordinates is given by

$$(z_1, z_2) = \Big(\prod_{i=1}^{m-1} F_i^{\alpha_i}, \prod_{i=2}^{m} F_i^{\beta_i}\Big)$$

and the system is integrable with first integral $z_1^q z_2^p$.

Case II: The system is integrable with first integral $\Phi(x,y) \sim x^q y^p$ and there exist analytic functions $F_1(x,y), \cdots, F_m(x,y), K_1(x,y), \cdots, K_m(x,y)$ defined in a neighborhood of the origin and numbers $\alpha_1, \cdots, \alpha_m$ satisfying $\dfrac{\partial F_i}{\partial x}\dot{x} + \dfrac{\partial F_i}{\partial y}\dot{y} = K_i F_i$ and

(i) $F_1(x,y) = x + o(x,y),\ F_i(0,0) = 1$ for $i = 2, \cdots, m$.

(ii) $\sum\limits_{i=1}^{m} \alpha_i K_i = p$.

The linearizing change of coordinates is given by

$$(z_1, z_2) = \Big(\prod_{i=1}^{m} F_i^{\alpha_i},\ (\Phi/z_1^q)^{1/p}\Big).$$

Case III: The system is integrable with first integral $\Phi(x,y) \sim x^q y^p$ and there exist analytic functions $F_1(x,y) \cdots, F_m(x,y), K_1(x,y), \cdots, K_m(x,y)$ defined in a neighborhood of the origin and numbers $\beta_1, \cdots, \beta_m$ satisfying $\dfrac{\partial F_i}{\partial x}\dot{x} + \dfrac{\partial F_i}{\partial y}\dot{y} = K_i F_i$ and

(i) $F_1(x,y) = y + o(x,y),\ F_i(0,0) = 1$ for $i = 2, \cdots, m$.

(ii) $\sum\limits_{i=1}^{m} \beta_i K_i = -q$.

The linearizing change of coordinates is given by

$$(z_1, z_2) = \Big((\Phi/z_2^p)^{1/q},\ \prod_{i=1}^{m} F_i^{\beta_i}\Big).$$

In fact, this theorem tells us that, if system (3) is integrable and one of the equations is linearizable, then the other equation should be linearizable too. More details on the Darboux method of integration and linearization can be found in [1].

3 The linearizability conditions

In this section, we will find the necessary and sufficient conditions for linearizability of system (9). Using a straightforward modification of Mathematica code from [8], we have computed the first six pairs of linearizability quantities. The polynomials are too long, so we do not present them here. The interested reader can easily compute them using any available computer algebra system with the algorithms of [1], for instance. Then, we find that $V(I_4) = V(I_3)$. Using the routine minAssGTZ of the computer algebra system SINGULAR, we obtain the minimal associated primes of the ideal I_3. Finally, for each component we find the linearizing transformation (6). Thus, we obtain the necessary and sufficient conditions for linearizability of system (9) as follows.

Theorem 4 System (9) is linearizable if and only if one of the following conditions holds:

$$(1) b_{30} = 4a_{30}, a_{21} = a_{12} = a_{03} = b_{21} = b_{12} = 0.$$
$$(2) b_{30} = \frac{7a_{30}}{2}, a_{21} = a_{12} = a_{03} = b_{21} = b_{03} = 0.$$
$$(3) a_{21} = a_{12} = a_{03} = b_{21} = b_{12} = b_{03} = 0.$$
$$(4) a_{21} = a_{12} = b_{30} = b_{21} = b_{12} = 0.$$
$$(5) b_{30} = 3a_{30}, a_{21} = a_{12} = b_{21} = b_{12} = 0.$$
$$(6) b_{30} = a_{30}, a_{21} = a_{12} = b_{21} = b_{12} = 0.$$
$$(7) a_{30} = 2b_{30}, a_{21} = a_{03} = b_{21} = b_{03} = 0.$$
$$(8) b_{30} = -a_{30}, b_{21} = -a_{21}, b_{12} = -a_{12}, b_{03} = -a_{03}.$$
$$(9) a_{30} = b_{30} = 0.$$
$$(10) b_{30} = 2a_{30}, a_{21} = 0.$$

Proof First, according to the definition of linearizability quantities, we calculate the first six groups of linearizability quantities,

$$i_k = c_{5k,k}, j_k = d_{5k,k}, k = 1,2,3,4,5,6,$$

where

$$i_1 = \frac{1}{3}a_{21}(a_{30} + b_{30}),\ j_1 = \frac{1}{3}(3a_{21}b_{30} - 2a_{30}b_{21} + b_{21}b_{30}),$$

$$i_2 = \frac{1}{81}(-198a_{21}^2a_{30}^2 - 4a_{12}a_{30}^3 - 18a_{21}a_{30}^2b_{21} - 117a_{21}^2a_{30}b_{30} + 6a_{12}a_{30}^2b_{30} + 39a_{21}a_{30}b_{21}b_{30} + 18a_{21}^2b_{30}^2 + 6a_{12}a_{30}b_{30}^2 - 6a_{21}b_{21}b_{30}^2 - 4a_{12}b_{30}^3),$$

$$j_2 = \frac{1}{81}(-14a_{30}^3b_{12} - 84a_{21}a_{30}^2b_{21} + 84a_{30}^2b_{21}^2 + 216a_{21}^2a_{30}b_{30} + 18a_{12}a_{30}^2b_{30} + 39a_{30}^2b_{12}b_{30} + 9a_{21}a_{30}b_{21}b_{30} - 90a_{30}b_{21}^2b_{30} - 72a_{21}^2b_{30}^2 - 45a_{12}a_{30}b_{30}^2 - 24a_{30}b_{12}b_{30}^2 + 3a_{21}b_{21}b_{30}^2 + 24b_{21}^2b_{30}^2 + 18a_{12}b_{30}^3 + 4b_{12}b_{30}^3).$$

We omit the polynomials $i_3, j_3, i_4, j_4, i_5, j_5, i_6, j_6$ because they are too long. By the relationship between the ideal and variety, we find that $V(I_4) = V(I_3)$. Then making use of the routine minAssGTZ, we find the irreducible decomposition of the variety of the ideal $I3 =< i1, j1, i2, j2, i3, j3 >$ over the field of rational numbers, and- the obtained decomposition consists of 10 components defined by the prime ideals listed in the theorem. That is to say, that at least one of the above ten conditions hold is a necessary condition for the system (9) to be linearizable.

We now show that under each condition the system (9) is linearizable.

Case 1 In this case, system (9) can be written as

$$\begin{aligned} \dot{x} &= x(1 - a_{30}x^3), \\ \dot{y} &= y(-5 + 4a_{30}x^3 + b_{03}y^3). \end{aligned} \tag{15}$$

Using the described Darboux integrability approach we have found that the system has the following four algebraic invariant curves: $l_1 = x,\ l_2 = y, l_3 = 1 - a_{30}x^3$ and $l_4 = 1 - a_{30}x^3 - \frac{b_{03}y^3}{5}$ with corresponding cofactors

$$\begin{aligned} k_1 &= 1 - a_{30}x^3, \quad k_2 = -5 + 4a_{30}x^3 + b_{03}y^3, \\ k_3 &= -3a_{30}x^3, \quad\ \ k_4 = -3a_{30}x^3 + 3b_{03}y^3. \end{aligned}$$

Solving equations $k_1 + \alpha_3k_3 + \alpha_4k_4 = 1$, $k_2 + \beta_3k_3 + \beta_4k_4 = -5$, we have $\alpha_3 = -\frac{1}{3}$, $\alpha_4 = 0$, $\beta_3 = \frac{5}{3}$ and $\beta_4 = -\frac{1}{3}$. By Theorem 3, a Darboux linearization of (15) is given by

$$z_1 = xl_3^{-\frac{1}{3}},\ z_2 = yl_3^{\frac{5}{3}}l_4^{-\frac{1}{3}}.$$

Case 2 In this case, system (9) takes the form

$$\begin{aligned} \dot{x} &= x(1 - a_{30}x^3), \\ \dot{y} &= y\Big(-5 + \frac{7a_{30}}{2}x^3 + b_{12}xy^2\Big). \end{aligned} \tag{16}$$

It has four algebraic invariant curves: $l_1 = x,\ l_2 = y,\ l_3 = 1 - a_{30}x^3$ and $l_4 = 1 - a_{30}x^3 - \frac{2}{9}b_{12}xy^2$ with corresponding cofactors

$$\begin{aligned} &k_1 = 1 - a_{30}x^3, \quad k_2 = -5 + \frac{7a_{30}}{2}x^3 + b_{12}xy^2, \\ &k_3 = -3a_{30}x^3, \qquad k_4 = -3a_{30}x^3 + 2b_{12}xy^2. \end{aligned}$$

Thus, a Darboux linearization of (16) is given by

$$z_1 = xl_3^{-\frac{1}{3}},\ z_2 = yl_3^{\frac{5}{3}}l_4^{-\frac{1}{2}}.$$

Case 3 In this case, the system (9) has the form

$$\begin{aligned} \dot{x} &= x(1 - a_{30}x^3), \\ \dot{y} &= y(-5 + b_{30}x^3) \end{aligned} \tag{17}$$

and admits invariant curves $l_1 = x,\ l_2 = y$ and $l_3 = 1 - a_{30}x^3$, with corresponding cofactors $k_1 = 1 - a_{30}x^3,\ k_2 = -5 + b_{30}x^3,$ and $k_3 = -3a_{30}x^3$. A Darboux linearization is given by

$$z_1 = xl_3^{-\frac{1}{3}},\ z_2 = yl_3^{\frac{b_{30}}{3a_{30}}}.$$

Case 4 In this case, the corresponding system is

$$\begin{aligned} \dot{x} &= x(1 - a_{30}x^3 - a_{03}y^3), \\ \dot{y} &= y(-5 + b_{03}y^3). \end{aligned} \tag{18}$$

It has 3 invariant curves $l_1 = x,\ l_2 = y$ and $l_3 = 1 - \frac{b_{03}y^3}{5}$ with corresponding cofactors $k_1 = 1 - a_{30}x^3 - a_{03}y^3,\ k_2 = -5 + b_{03}y^3$ and $k_3 = 3b_{03}y^3$. The linearizing transformations for the second equation of (18) is given by $z_2 = yl_3^{-\frac{1}{3}}$. But it is hard to find a linearization for the first equation of (18).

However, from $\alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \mathrm{div} = 0$, where div represent the divergence of system (18), one can obtain $\alpha_1 = -4, \alpha_2 = -\frac{8}{5}, \alpha_3 = \frac{5a_{03} - 4b_{03}}{5b_{03}}$. So by Theorem 2, the system (18) has an integrating factor $\mu = x^{-4}y^{-\frac{8}{5}}l_3^{\frac{5a_{03}-4b_{03}}{5b_{03}}}$ and then we can find a first integral of the form $\Psi(x,y) = x^5y + h.o.t.$ Hence, the linearizing transformation for the first equation of (18) is

$$z_1 = \Big[\frac{\Psi(x,y)}{z_2}\Big]^{\frac{1}{5}}.$$

Case 5 In this case, system (9) takes the form

$$\begin{aligned} \dot{x} &= x(1 - a_{30}x^3 - a_{03}y^3), \\ \dot{y} &= y(-5 + 3a_{30}x^3 + b_{03}y^3). \end{aligned} \tag{19}$$

Using the transformations

$$\begin{aligned} X &= g(x,y) = x^3, \\ Y &= h(x,y) = y^3, \\ T &= 3t, \end{aligned} \tag{20}$$

we rewrite system (19) into the form

$$\begin{aligned}\dot{X} &= X(1-a_{30}X-a_{03}Y),\\ \dot{Y} &= Y(-5+3a_{30}X+b_{03}Y).\end{aligned} \tag{21}$$

By Theorem 1, system (21) is linearizable. Therefore, system (19) is also linearizable.

Case 6 In this case, system (9) takes the form

$$\begin{aligned}\dot{x} &= x(1-a_{30}x^3-a_{03}y^3),\\ \dot{y} &= y(-5+a_{30}x^3+b_{03}y^3),\end{aligned} \tag{22}$$

Using the transformations (20), we rewrite system (22) into the form

$$\begin{aligned}\dot{X} &= X(1-a_{30}X-a_{03}Y),\\ \dot{Y} &= Y(-5+a_{30}X+b_{03}Y).\end{aligned} \tag{23}$$

By Theorem 1, system (23) is linearizable. So system (22) is also linearizable.

Case 7 In this case, system (9) takes the form

$$\begin{aligned}\dot{x} &= x(1-2b_{30}x^3-a_{12}xy^2),\\ \dot{y} &= y(-5+b_{30}x^3+b_{12}xy^2).\end{aligned} \tag{24}$$

It admits three algebraic invariant curves $l_1=x$, $l_2=y$ and $l_3=1+\frac{1}{9}(a_{12}-2b_{12})xy^2$ with corresponding cofactors $k_1=1-2b_{30}x^3-a_{12}xy^2$, $k_2=-5+b_{30}x^3+b_{12}xy^2$ and $k_3=(2b_{12}-a_{12})xy^2$.

These three invariant curves are not sufficient to directly obtain the linearizable transformation of the system (24). However, from $\alpha_1k_1+\alpha_2k_2+\alpha_3k_3+\mathrm{div}=0$, where div represents the divergence of system (24), we obtained $\alpha_1=\frac{13}{3},\alpha_2=\frac{5}{3},\alpha_3=\frac{4b_{12}-7a_{12}}{3(a_{12}+2b_{12})}$. So, by Theorem 2 the system (24) has an integral factor $\mu=l_1^{\frac{13}{3}}l_2^{\frac{5}{3}}l_3^{\frac{4b_{12}-7a_{12}}{3(a_{12}+2b_{12})}}$, and we can find a first integral of the form $\Psi(x,y)=x^5y+h.o.t.$.

Taking $l_4=\frac{\Psi(x,y)}{x^5y}=1+h.o.t.$. Then l_4 is an invariant curve of system (24) with corresponding cofactor

$$k_4=9b_{30}x^3-(5a_{12}+b_{12})xy^2.$$

Thus, a Darboux linearization of the system (24) is give by

$$z_1=xl_3^{-\frac{1}{9}}l_4^{-\frac{2}{9}},\ z_2=yl_3^{-\frac{5}{9}}l_4^{-\frac{1}{9}}.$$

Case 8 In this case, system (9) takes the form

$$\begin{aligned}\dot{x} &= x(1-a_{30}x^3-a_{21}x^2y-a_{12}xy^2-a_{03}y^3),\\ \dot{y} &= y(-5-a_{30}x^3-a_{21}x^2y-a_{12}xy^2-a_{03}y^3),\end{aligned} \tag{25}$$

and admits three invariant curves $l_1=x$, $l_2=y$ and

$$l_3=1-a_{30}x^3+a_{21}x^2y+\frac{1}{3}a_{12}xy^2+\frac{1}{5}a_{03}y^3$$

with corresponding cofactors $k_1=1-\xi$, $k_2=-5-\xi$ and $k_3=-3\xi$ with

$$\xi=a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3.$$

Thus, a Darboux linearization is given by

$$z_1=xl_3^{-\frac{1}{3}},\ z_2=yl_3^{-\frac{1}{3}}.$$

Case 9 In this case, system (9) takes the form

$$\begin{aligned} \dot{x} &= x(1 - a_{21}x^2y - a_{12}xy^2 - a_{03}y^3), \\ \dot{y} &= y(-5 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3). \end{aligned} \tag{26}$$

However, it is hard to find more invariant curve except x and y. Now, we try to prove it by finding formal linearization z_1 and z_2.

Firstly, we look for a Darboux first integral of system (26) in the form of a power series

$$F(x,y) = \sum_{k=1}^{\infty} f_k(x)y^k, \tag{27}$$

satisfying $f_1(x) = x^5 + o(x^5)$. According to $\dot{F} = 0$, the functions $f_k(x)$ satisfy the system of equations

$$\begin{aligned} &xf_1'(x) - 5f_1(x) = 0, \quad f_1(x) = x^5 + o(x^5), \\ &xf_k'(x) - 5kf_k(x) = F_k(x), \quad f_k(x) = o(x^5), \quad k = 2, 3, \cdots, \end{aligned}$$

where

$$\begin{aligned} &F_2(x) = a_{21}x^3f_1'(x) - b_{21}x^2f_1(x), \\ &F_3(x) = a_{21}x^3f_2'(x) + a_{12}x^2f_1'(x) - 2b_{21}x^2f_2(x) - b_{12}xf_1(x), \\ &F_k(x) = a_{21}x^3f_{k-1}' + a_{12}x^2f_{k-2}' + a_{03}xf_{k-3}' - (k-1)b_{21}x^2f_{k-1} - (k-2)b_{12}xf_{k-2} - (k-3)b_{03}f_{k-3} \end{aligned}$$

for $k > 3$. The first of them has the solution $f_1(x) = x^5$, which results in $F_2(x) = (5a_{21} - b_{21})x^7$. Then, from the second equation we have $f_2(x) = \dfrac{b_{21} - 5a_{21}}{3}x^7$, which results in

$$F_3(x) = (5a_{12} - b_{12})x^6 - \frac{1}{3}(7a_{21} - 2b_{21})(5a_{21} - b_{21})x^7.$$

Step by step, we can obtain $f_k(x)$ easily. Obviously, $f_k(x)$ is a polynomial of degree $2k+3$ for any $k = 1, 2, \cdots$. Therefore, the system is integrable.

By Theorem 3, we only need to prove that one of the linearizing transformation z_1 or z_2 exists. Now, we look for a linearizable transformation of the second equation of system (26) in the form of a power series

$$z_2 = \sum_{k=1}^{\infty} g_k(x)y^k, \tag{28}$$

satisfying $g_1(x) = 1 + O(x)$. According to $\dot{z}_2 = -5z_2$, the functions $g_k(x)$ satisfy the system of equations

$$\begin{aligned} &g_1'(x) = 0, \quad g_1(0) = 1, \\ &g_k'(x) - 5(k-1)g_k(x) = G_k(x), \quad g_k(0) = 0, \quad k = 2, 3, \cdots, \end{aligned}$$

where

$$\begin{aligned} &G_2(x) = a_{21}x^3g_1' - b_{21}x^2g_1, \\ &G_3(x) = a_{21}x^3g_2' + a_{12}x^2g_1' - 2b_{21}x^2g_2 - b_{12}xg_1, \\ &G_k(x) = a_{21}x^3g_{k-1}' + a_{12}x^2g_{k-2}' + a_{03}xg_{k-3}' - (k-1)b_{21}x^2g_{k-1} - (k-2)b_{12}xg_{k-2} - (k-3)b_{03}g_{k-3} \end{aligned}$$

for $k > 3$. The first of them has the solution $g_1(x) = 1$, which results in $G_2(x) = -b_{21}x^2$. Then, from the second equation we have

$$g_2(x) = \frac{1}{125}b_{21}(25x^2 + 10x + 2),$$

which results in

$$G_3(x) = \frac{2}{5}(a_{21} - b_{21})b_{21}x^4 + \frac{2}{25}(a_{21} - 2b_{21})b_{21}x^3 - \frac{4}{125}b_{21}^2x^2 - b_{12}x.$$

Step by step, we can obtain $g_n(x)$ easily. Obviously, both $G_k(x)$ and $g_k(x)$ are polynomials of degree $2(k-1)$. By Theorem 3 the system is linearizable.

Case 10 In this case, system (9) takes the form

$$\begin{aligned} \dot{x} &= x(1 - a_{30}x^3 - a_{12}xy^2 - a_{03}y^3), \\ \dot{y} &= y(-5 + 2a_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3). \end{aligned} \tag{29}$$

Similarly as in case (9), we look for a first integral $F(x, y)$ and a linearizable transformation z_2 in the form of a power series,

$$F(x, y) = \sum_{k=1}^{\infty} f_k(x)y^k,\ z_2 = \sum_{k=1}^{\infty} g_k(x)y^k,$$

which satisfy $\dot{F} = 0$, $\dot{z}_2 = -5z_2$, $f_1(x) = x^5 + o(x^5)$ and $g_1(0) = 1$.

So the functions $f_k(y)$ and $g_k(x)$ satisfy the system of equations

$$\begin{aligned} &(1 - a_{30}x^3)xf_k'(x) + k(2a_{30}x^3 - 5)f_k(x) = F_k(x),\ f_1(x) = x^5 + o(x^5),\ \ f_k(x) = o(x^5), \\ &(1 - a_{30}x^3)xg_k' + (2ka_{30}x^3 - 5k + 5)g_k = G_k(x),\ g_1(0) = 1,\ \ g_k(0) = 0,\ \ k = 2, 3, \cdots, \end{aligned}$$

where

$$\begin{aligned} F_k(x) &= a_{12}x^2f_{k-2}' + a_{03}xf_{k-3}' - (k-1)b_{21}x^2f_{k-1} - (k-2)b_{12}xf_{k-2} - (k-3)b_{03}h_{k-3}, \\ G_k(x) &= a_{12}x^2g_{k-2}' + a_{03}xg_{k-3}' - (k-1)b_{21}x^2g_{k-1} - (k-2)b_{12}xg_{k-2} - (k-3)b_{03}g_{k-3}, \end{aligned}$$

and $g_j(x) = h_j(x) = 0$ for $j \leqslant 0$.

Step by step, we can obtain from the above equations that

$$\begin{aligned} f_1(x) &= \frac{x^5}{1 - a_{30}x^3},\ \ f_2(x) = \frac{b_{21}x^7}{3(1 - a_{30}x^3)^2}, \\ f_3(x) &= \frac{1}{1260(1 - a_{30}x^3)^3}(700a_{12}x^6 - 315b_{21}x^7 - 60b_{12}b_{21}x^8 - 420a_{12}a_{30}x^9 + 504a_{30}b_{21}x^{10}), \\ &\cdots \\ g_1(x) &= (1 - a_{30}x^3)^{2/3},\ \ g_2(x) = -\frac{1}{3}b_{21}x^2(1 - a_{30}x^3)^{-1/3}, \\ g_3(x) &= \frac{1}{420}(105b_{21}x^2 + 20b_{12}b_{21}x^3 + 140a_{12}a_{30}x^4 - 168a_{30}b_{21}x^5)(1 - a_{30}x^3)^{-4/3}, \\ &\cdots \end{aligned}$$

Therefore, by Theorem 3 the system (29) is linearizable.

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