Boundary feedback stabilization for a class of distributed-order fractional reaction diffusion systems

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Abstract: In this paper, the problem of boundary feedback stabilization for a class of distributed-order fractional reaction diffusion systems is studied. Firstly, the stability of the target system is analyzed. By using the backstepping method, the explicit kernel matrix function is obtained, and a new state feedback controller is designed to stabilize the original system. Then we consider two kinds of output measurements, that is, the non-collocated output and the collocated output. In the two cases, the corresponding observers and output feedback controllers are designed, respectively, to achieve the asymptotical stability of the studied system.

Key words: distributed-order fractional system; reaction diffusion system; backstepping transformation; stability; boundary control

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1 Introduction

The fractional calculus has allowed the operation of integration and differentiation to any order. The order may take any real or imaginary number\(^{[1-4]}\). It has been shown that there are an increasing number of practical systems whose behaviors can be described more accurately than those of the fractional order systems (or systems containing fractional derivatives and integrals)\(^{[5-6]}\). Moreover, fractional calculus has also been applied to many directions of the control theory and their applications\(^{[2-3,7-9]}\).

The idea of the distributed-order equation was first proposed by CAPUTO in 1969\(^{[10]}\) and later developed by CAPUTO himself\(^{[11-12]}\) and BAGLEY et al\(^{[13-14]}\). These distributed-order equations were introduced in the constitutive equations of ultra slow diffusion\(^{[15]}\). Since then, the basic theory about the solutions of this kind of equations has attracted many researchers’ attention, and some valuable and meaningful results have been obtained\(^{[16-18]}\). Meanwhile, various applications of this kind of equations have appeared, including \(^{[19-24]}\). In \(^{[25]}\), the authors provided an introduction of the research results about the distributed-order dynamic system and control.

In \(^{[26]}\), for the first time, the authors attempted to study the boundary stabilization for fractional partial differential equations. By using the numerical method, the author designed a feedback controller to stabilize a fractional wave equation. In \(^{[27]}\), the stabilization of fractional wave equations subject to the delayed measurement at the boundary was considered, and again, the stability test was carried out by the numerical method. It should be pointed out that, in \(^{[26-27]}\), there is no rigorous mathematical proof. Recently, ZHOU et al\(^{[28]}\) investigated boundary feedback stabilization for unstable time fractional reaction diffusion equations and designed new state feedback controllers with actuation by the backstepping method. However, as far as we know, there are few contributions to boundary feedback stabilization for the distributed-order fractional systems (DOFSs)\(^{[29]}\). Motivated by this, in this paper, we try to study boundary feedback stabilization for a class of unstable DOFSs by the backstepping method.

We consider the distributed-order fractional equations as follows:

\[
\begin{aligned}
\frac{C}{0}D_t^{\omega(\alpha)} W (t, x) &= AW_{xx} (t, x) + BW (t, x), \ t \geq 0, \ x \in (0, 1), \\
W (t, 0) &= \theta, \ t \geq 0, \\
W (t, 1) &= U (t), \ t \geq 0, \\
W (0, x) &= W_0 (x), \ x \in [0, 1], \\
Y_{\text{out}} (t) &= W_x (t, 0), \ \text{or} \ Y_{\text{out}} (t) = W_x (t, 1), \ t \geq 0,
\end{aligned}
\]

where \( W (t, x) = [W_1 (t, x), W_2 (t, x), \ldots, W_n (t, x)]^T \in \mathbb{R}^n \) is the state vector of the system, \( \theta \) denotes the zero vector in \( \mathbb{R}^n \), \( A, B \in \mathbb{R}^{n \times n} \) are coefficient matrices, \( Y_{\text{out}} \) is the output, and \( U(t) \in \mathbb{R}^n \) is the control input. The
The objective of this paper is to design boundary feedback controls to stabilize system (1) which, in some cases, can be unstable without control.

The rest of this paper is organized in the following way: in the next section, some necessary definitions and useful lemmas are introduced. In section 3, by employing the backstepping method, a state feedback controller is designed to achieve the stabilization of the original system. Section 4 and section 5 focus on the design of the non-collocated output and collocated output controllers which can realize the asymptotic stability of the closed-loop systems. Finally, an example is given to illustrate our results.

2 Preliminaries

We first present some basic definitions and facts to be applied throughout this paper.

Definition 1 Let \( f(t) : [0, \infty) \rightarrow \mathbb{R} \) be absolutely continuous and \( 0 < \alpha < 1 \). The Caputo fractional derivative of order \( \alpha \) of \( f(t) \) is defined by

\[
\frac{C}{0} D_t^\alpha f(t) \triangleq \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} f'(s) \, ds, \quad t \geq 0.
\]  

(2)

Definition 2 \[18\] The distributed-order fractional differential operator in the Caputo sense with respect to an order density function (weight function) \( w(\alpha) \geq 0 \) is defined as

\[
\frac{C}{0} D_t^{w(\alpha)} x(t) = \int_0^1 w(\alpha) \frac{C}{0} D_t^\alpha x(t) \, d\alpha,
\]

(3)

where \( w(\alpha) \) is an integrable function on the interval \([0, 1]\).

When \( w(\alpha) = \delta(\alpha - \alpha_0) \), (3) converts to (2). Here, \( \delta(\cdot) \) is the well known Dirac Delta function.

The next lemma gives an important property of the Caputo distributed-order fractional derivative, which is helpful in constructing appropriate Lyapunov functions.

Lemma 1 \[18\] Let \( x(t) : [0, \infty) \rightarrow \mathbb{R}^n \) be a continuously differentiable function. Then for any time \( t \geq 0 \), the following relationship holds:

\[
\frac{1}{2} \frac{C}{0} D_t^{w(\alpha)} \left\{ x^\top(t) x(t) \right\} \leq x^\top(t) \frac{C}{0} D_t^{w(\alpha)} x(t).
\]

Lemma 2 (Poincaré inequality) Let \( Z(t, x) : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}^n \) be continuously differentiable with respect to \( x \). Then for any time \( t \geq 0 \),

\[
\int_0^1 Z^\top(t, x)Z(t, x) \, dx \leq 2Z^\top(t, 1)Z(t, 1) + 4 \int_0^1 Z_x^\top(t, x)Z_x(t, x) \, dx.
\]

Proof Using the method of integral by parts and in view of Young’s inequality, for any \( t \geq 0 \), we have

\[
\int_0^1 Z^\top(t, x)Z(t, x) \, dx = xZ^\top(t, x)Z(t, x) \bigg|_0^1 - 2 \int_0^1 xZ_x^\top(t, x)Z(t, x) \, dx
\]
coefficient matrices in (1) and (4).

positive real parts.

Here which leads to the following conclusion.

The final value theorem tells us that

\[ G(t) = \int_0^t F_1(s) \, ds, \quad t \geq 0, \] where \( G(t) = \mathcal{L}\{G(t)\}, s \in \mathbb{C}. \) Then

\[ \hat{G}(s) = \frac{F_1(s)}{s} = \frac{1}{s \left( \int_0^1 w(\alpha) s^\alpha d\alpha + \mu \right)}, \quad s \in \mathbb{C}. \]

The final value theorem tells us that

\[ \lim_{t \to \infty} G(t) = \lim_{s \to 0} s \hat{G}(s) = \frac{1}{\mu}, \]

which leads to the following conclusion.

Lemma 3 Let \( Q(t) = 1 - \mu \int_0^t F_1(\tau) d\tau, \quad t \geq 0. \) Then \( \lim_{t \to \infty} Q(t) = 0. \)

The following lemma gives the existence and uniqueness of the solution to (4) and its asymptotic stability.

Lemma 4 For any \( Z_0 \in [L^2(0,1)]^n, \) (4) has a unique solution \( Z(t,x) \in C\left(0, \infty; [L^2(0,1)]^n\right). \) In addition,

\[ \|Z(t,\cdot)\|_{[L^2(0,1)]^n}^2 \leq Q(t) \times \|Z_0\|_{[L^2(0,1)]^n}^2, \quad t \geq 0. \] (6)
It follows from (8) and (9) that
\[ V(t) = \frac{1}{2} \int_0^1 Z^\top(t, x) Z(t, x) \, dx, \quad t \geq 0. \] (7)

According to lemma 1, we have
\[
\begin{align*}
\frac{C}{0} D_t^{w(\alpha)} V(t) & \leq \int_0^1 Z^\top(t, x) \frac{C}{0} D_t^{w(\alpha)} Z(t, x) \, dx = \int_0^1 Z^\top(t, x) AZ_x(t, x) \, dx - \int_0^1 Z^\top(t, x) CZ(t, x) \, dx \\
& = Z^\top(t, x) AZ_x(t, x) \Big|_0^1 - \int_0^1 Z^\top_x(t, x) AZ_x(t, x) \, dx - \int_0^1 Z^\top(t, x) CZ(t, x) \, dx \\
& = - \int_0^1 Z_x^\top(t, x) AZ_x(t, x) \, dx - \int_0^1 Z^\top(t, x) CZ(t, x) \, dx.
\end{align*}
\] (8)

From lemma 2, we get
\[
\int_0^1 Z_x^\top(t, x) AZ_x(t, x) \, dx \geq \lambda_{\min} \left( \frac{A^\top + A}{2} \right) \int_0^1 Z_x^\top(t, x) Z_x(t, x) \, dx \\
\geq \frac{1}{4} \lambda_{\min} \left( \frac{A^\top + A}{2} \right) \int_0^1 Z^\top(t, x) Z(t, x) \, dx.
\] (9)

It follows from (8) and (9) that
\[
\frac{C}{0} D_t^{w(\alpha)} V(t) \leq - \frac{1}{4} \lambda_{\min} \left( \frac{A^\top + A}{2} \right) \int_0^1 Z^\top(t, x) Z(t, x) \, dx - \lambda_{\min} \left( \frac{C^\top + C}{2} \right) \int_0^1 Z^\top(t, x) Z(t, x) \, dx \\
\leq - 2 \left[ \frac{1}{4} \lambda_{\min} \left( \frac{A^\top + A}{2} \right) + \lambda_{\min} \left( \frac{C^\top + C}{2} \right) \right] V(t) \triangleq - \mu V(t).
\]

Hence, there exists a nonnegative function \( M(t) \) satisfying
\[
\frac{C}{0} D_t^{w(\alpha)} V(t) + M(t) = - \mu V(t), \quad t \geq 0.
\] (10)

By taking the Laplace transform on both sides of (10), we get
\[
\left( \int_0^1 w(\alpha) s^\alpha d\alpha \right) \tilde{V}(s) = \left( \int_0^1 w(\alpha) s^{\alpha - 1} d\alpha \right) V(0) + \tilde{M}(s) = - \mu \tilde{V}(s), \quad s \in \mathbb{C},
\] (11)

where \( \tilde{V}(s), \tilde{M}(s) \) denote the Laplace transforms of the functions \( V(t) \) and \( M(t) \). It follows from (11) that
\[
\tilde{V}(s) = \frac{V(0)}{s} - \frac{\mu V(0)}{s} \tilde{F}_1(s) - \tilde{M}(s) \tilde{F}_1(s).
\] (12)

By taking the inverse Laplace transform on both sides of (12), we obtain
\[
V(t) = V(0) - \mu V(0) \int_0^t F_1(\tau) d\tau - \int_0^t M(\tau) F_1(t - \tau) d\tau \\
\leq \left( 1 - \mu \int_0^t F_1(\tau) d\tau \right) V(0) + V(0) \times Q(t), \quad t \geq 0,
\]

which, together with (7), means that (6) holds. From lemma 3, we can conclude
\[
\lim_{t \to \infty} \| Z(t, \cdot) \|_{L^2(0, 1)^n}^2 = 0.
\]
Substituting (15) and (16) into (4), it can be seen that the matrix value function
which can transform the system (1) into the target system (4). Here \(K(x, y) : \{(x, y) | 0 \leq y \leq x \leq 1\} \rightarrow \mathbb{R}^{n \times n}\) is a matrix value function to be determined.

In (13), calculating the \(w^{(\alpha)}\)th order distributed-order fractional derivative with respect to \(t\), we have

\[
\frac{C}{0} D_{t}^{w(\alpha)} Z(t, x) = \frac{C}{0} D_{t}^{w(\alpha)} W(t, x) - \int_{0}^{x} K(x, y) \frac{C}{0} D_{t}^{w(\alpha)} W(t, y) dy.
\] (14)

Substituting (1) into (14) and integrating by parts, we have

\[
\frac{C}{0} D_{t}^{w(\alpha)} Z(t, x) = AW_{xx}(t, x) + BW(t, x) - \int_{0}^{x} K(x, y) [AW_{yy}(t, y) + BW(t, y)] dy
\]

\[
= AW_{xx}(t, x) + BW(t, x) - K(x, x) AW_{x}(t, x) + K(x, 0) AW_{x}(t, 0)
\]

\[
+ K_{y}(x, x) AW(t, x) - K_{y}(x, 0) AW(t, 0)
\]

\[
- \int_{0}^{x} [K_{yy}(x, y) AW(t, y) + K(x, y) BW(t, y)] dy.
\] (15)

Note that

\[
Z_{xx}(t, x) = W_{xx}(t, x) - \left[ \frac{d}{dx}(K(x, x)) + K_{x}(x, x) \right] W(t, x)
\]

\[
- K(x, x) W_{x}(t, x) - \int_{0}^{x} K_{xx}(x, y) W(t, y) dy.
\] (16)

Substituting (15) and (16) into (4), it can be seen that the matrix value function \(K(x, y)\) must satisfy

\[
\begin{align*}
AK_{xx}(x, y) - K_{yy}(x, y) A & = K(x, y) B + CK(x, y), \\
AK'(x, x) + AK_{x}(x, x) + K_{y}(x, x) A & = -B - C, \\
K(x, x) A - AK(x, x) & = O, \\
K(x, 0) & = O,
\end{align*}
\] (17)

where \(O \in \mathbb{R}^{n \times n}\) is the zero matrix. The expression of solution to (17) is given in the following lemma.

**Lemma 5** Let (H) hold. Then the solution \(K(x, y)\) to (17) is explicitly given by

\[
K(x, y) = - \sum_{n=0}^{\infty} \frac{y(x^2 - y^2)^n [A^{-1}(B + C)]^{n+1}}{n!(n + 1)!2^{2n+1}}, \quad 0 \leq y \leq x \leq 1.
\] (18)

**Proof** It suffices to check that (18) satisfies each equation of (17). Indeed, from (18), we get

\[
K_{x}(x, y) = - \sum_{n=1}^{\infty} \frac{xy(x^2 - y^2)^{n-1} [A^{-1}(B + C)]^{n+1}}{(n-1)!(n + 1)!2^{2n}},
\] (19)
\[ K_{xx}(x, y) = - \sum_{n=1}^{\infty} \frac{y(x^2 - y^2)^{n-1}[A^{-1}(B + C)]^{n+1}}{(n-1)!(n+1)!2^{2n}} - \sum_{n=2}^{\infty} \frac{x^2y(x^2 - y^2)^{n-2}[A^{-1}(B + C)]^{n+1}}{(n-2)!(n+1)!2^{2n-1}}, \]  

and

\[ K_y(x, y) = \sum_{n=0}^{\infty} \frac{x(x^2 - y^2)^n[A^{-1}(B + C)]^{n+1}}{n!(n+1)!2^{2n+1}} + \sum_{n=1}^{\infty} \frac{y^2(x^2 - y^2)^{n-1}[A^{-1}(B + C)]^{n+1}}{(n-1)!(n+1)!2^{2n}}, \]  

Noting that \( AB = BA, AC = CA \) and \( BC = CB \), it follows from (20), (22) and (18) that

\[ AK_{xx}(x, y) - K_{yy}(x, y)A \]

\[ = - \sum_{n=1}^{\infty} \frac{y(x^2 - y^2)^{n-1}[A^{-1}(B + C)]^{n+1}}{(n-1)!(n+1)!2^{2n}} - \sum_{n=2}^{\infty} \frac{x^2y(x^2 - y^2)^{n-2}[A^{-1}(B + C)]^{n+1}}{(n-2)!(n+1)!2^{2n-1}} - \sum_{n=0}^{\infty} \frac{x(x^2 - y^2)^n[A^{-1}(B + C)]^{n+1}}{n!(n+1)!2^{2n+1}} + \sum_{n=1}^{\infty} \frac{y^2(x^2 - y^2)^{n-1}[A^{-1}(B + C)]^{n+1}}{(n-1)!(n+1)!2^{2n}} \]

\[ = - \left[ \frac{y[A^{-1}(B + C)]}{2} + \sum_{n=1}^{\infty} \frac{y(x^2 - y^2)^n[A^{-1}(B + C)]^{n+1}}{n!(n+1)!2^{2n+1}} \right] (B + C) \]

\[ = - \sum_{n=0}^{\infty} \frac{y(x^2 - y^2)^n[A^{-1}(B + C)]^{n+1}}{n!(n+1)!2^{2n+1}} (B + C) = K(x, y)B + CK(x, y), \]

which implies that \( K(x, y) \) satisfies the first equation in (17). Moreover, By (19) and (21), one has

\[ K_x(x, x) = - \frac{x^2[A^{-1}(B + C)]^2}{8} \]

and

\[ K_y(x, x) = - \frac{A^{-1}(B + C)}{2} + \frac{x^2[A^{-1}(B + C)]^2}{8}. \]

Then, we have

\[ AK'(x, x) + AK_x(x, x) + K_y(x, x)A = 2A \left[ K_x(x, x) + K_y(x, x) \right] = 2A \left\{ - \frac{x^2[A^{-1}(B + C)]^2}{8} - \frac{A^{-1}(B + C)}{2} + \frac{x^2[A^{-1}(B + C)]^2}{8} \right\} = -(B + C), \]

which shows that \( K(x, y) \) satisfies the second equation in (17). In addition, by direct calculation we can see that (18) satisfies the remaining equations in (17). The proof is complete.
Next, we seek the inverse transformation of (13). For any \( Z(t, x) \) satisfying (4), assume that

\[
W(t, x) = Z(t, x) + \int_0^t L(x, y)Z(t, y)dy.
\]

(23)

Here \( L(x, y) : \{(x, y) | 0 \leq y \leq x \leq 1\} \rightarrow \mathbb{R}^{n \times n} \) is a matrix value function to be determined. Once again, calculating the \( w(\alpha) \)th order distributed-order fractional derivative with respect to \( t \) on both sides of (23), we have

\[
\frac{C}{\alpha} D_t^{w(\alpha)} W(t, x) = \frac{C}{\alpha} D_t^{w(\alpha)} Z(t, x) + \int_0^t L(x, y)\frac{C}{\alpha} D_t^{w(\alpha)} Z(t, y)dy.
\]

(24)

Similar argument to the above, we know that the matrix value function \( L(x, y) \) should satisfy

\[
\begin{aligned}
AL_{xx}(x, y) - L_{yy}(x, y)A &= -BL(x, y) - L(x, y)C, \\
AL'(x, x) + AL_x(x, x) + L_y(x, x)A &= -B - C, \\
L(x, x)A - AL(x, x) &= O, \\
L(x, 0) &= O.
\end{aligned}
\]

(25)

Meanwhile, when (H) holds, the solution to (25) is given by

\[
L(x, y) = -y \sum_{n=0}^{\infty} (y^2 - x^2)^n [A^{-1}(B + C)]^{n+1} \frac{n!}{n!(n+1)!2^{n+1}}, \quad 0 \leq y \leq x \leq 1.
\]

(26)

**Remark 1** The invertibility of transformation (13) can be also shown by [32, theorem 3.1 and theorem 3.3]. However, the above argument is helpful to obtain the expression (26) of \( L(x, y) \) by using (25).

Now, we can propose the following state feedback control law:

\[
U(t) = \int_0^1 K(1, y)W(t, y)dy.
\]

(27)

With this state feedback control, the closed-loop system of (1) becomes

\[
\begin{aligned}
\frac{C}{\alpha} D_t^{w(\alpha)} W(t, x) &= AW_{xx}(t, x) + BW(t, x), \quad t \geq 0, \quad x \in (0, 1), \\
W(t, 0) &= \theta, \quad t \geq 0, \\
W(t, 1) &= \int_0^1 K(1, y)W(t, y)dy, \quad t \geq 0, \\
W(0, x) &= W_0(x), \quad x \in [0, 1].
\end{aligned}
\]

(28)

**Theorem 1** For any \( W_0 \in [L^2(0, 1)]^n \), system (28) has a unique solution \( W(t, x) \in C(0, \infty; [L^2(0, 1)]^n) \).

In addition, for some \( M > 0 \),

\[
\|W(t, \cdot)\|^2_{[L^2(0, 1)]^n} \leq MQ(t)\|W_0\|^2_{[L^2(0, 1)]^n}, \quad t \geq 0.
\]

**Proof** For any \( W(t, \cdot) \in [L^2(0, 1)]^n \), let \( F_1(W)(t, x) \triangleq Z(t, x) \), where \( Z(t, x) \) is defined by (13). Obviously, \( F_1 \) is a continuous linear operator from \([L^2(0, 1)]^n\) to itself. Hence \( F_1 \) is bounded, namely, there exists \( M_1 > 0 \) such that \( \|F_1\| \leq M_1 \). Then, for any \( W_0 \in [L^2(0, 1)]^n \),

\[
\|Z_0\|^2_{[L^2(0, 1)]^n} = \|F_1W_0\|^2_{[L^2(0, 1)]^n} \leq M_1^2 \|W_0\|^2_{[L^2(0, 1)]^n}.
\]
Conversely, for any \( Z(t, \cdot) \in [L^2(0, 1)]^n \), let \( F_2(Z)(t, x) \triangleq W(t, x) \), where \( W(t, x) \) is defined by (23). Similarly, we know that \( \| F_2 \| \leq M_2 \) for some \( M_2 > 0 \). Then

\[
\| W(t, \cdot) \|_{[L^2(0, 1)]^n}^2 = \| F_2 Z(t, \cdot) \|_{[L^2(0, 1)]^n}^2 \leq M_2^2 \| Z(t, \cdot) \|_{[L^2(0, 1)]^n}^2 .
\]

In view of lemma 4, we know that system (28) has a unique solution \( W(t, x) \). In addition,

\[
\| W(t, \cdot) \|_{[L^2(0, 1)]^n}^2 \leq M_2^2 \| Z(t, \cdot) \|_{[L^2(0, 1)]^n}^2 \leq M_2^2 Q(t) \| Z_0 \|_{[L^2(0, 1)]^n}^2 \leq M_2^2 Q(t) \| W_0 \|_{[L^2(0, 1)]^n}^2 , \quad t \geq 0,
\]

where \( M = M_2^2 M_2^2 \). The proof is complete.

4 Output feedback for non-collocated control

In this section, we try to design the non-collocated controller to asymptotically stabilize the system (1), that is, we consider \( W_x(t, 0) \) as the output measurement whereas the full state \( W(t, x) \) is unmeasurable and requires to be recovered. To this end, we first design an observer of (1) as

\[
\begin{align*}
\begin{cases}
\frac{d}{dt} \tilde{W}(t, x) = A \tilde{W}_x(t, x) + B \tilde{W}(t, x) + G(x) \left[ \tilde{W}_x(t, 0) - W_x(t, 0) \right], & t \geq 0, \; x \in (0, 1), \\
\tilde{W}(t, 0) = \theta, & t \geq 0, \\
\tilde{W}(t, 1) = U(t), & t \geq 0, \\
\tilde{W}(0, x) = \tilde{W}_0(x), & x \in [0, 1],
\end{cases}
\end{align*}
\]

(29)

where \( G(x) \) is the gain matrix of the observer to be determined later.

Let \( \tilde{W}(t, x) = \tilde{W}(t, x) - W(t, x) \) be the observer error. Then \( \tilde{W}(t, x) \) satisfies

\[
\begin{align*}
\begin{cases}
\frac{d}{dt} \tilde{W}(t, x) = A \tilde{W}_x(t, x) + B \tilde{W}(t, x) + G(x) \tilde{W}_x(t, 0), & t \geq 0, \; x \in (0, 1), \\
\tilde{W}(t, 0) = \theta, & t \geq 0, \\
\tilde{W}(t, 1) = \theta, & t \geq 0, \\
\tilde{W}(0, x) = \tilde{W}_0(x) - \tilde{W}_0(x) - W_0(x), & x \in [0, 1].
\end{cases}
\end{align*}
\]

(30)

To determine the gain matrix \( G(x) \) such that (30) is asymptotically stable, we consider the backstepping transformation: for any \( \tilde{W}(t, x) \) satisfying (30), assume that

\[
\tilde{Z}(t, x) = \tilde{W}(t, x) - \int_0^x P(x, y) \tilde{W}(t, y) dy,
\]

(31)

which will transform system (30) into an equivalent system,

\[
\begin{align*}
\begin{cases}
\frac{d}{dt} \tilde{Z}(t, x) = A \tilde{Z}_x(t, x) - C \tilde{Z}(t, x), & t \geq 0, \; x \in (0, 1), \\
\tilde{Z}(t, 0) = \theta, & t \geq 0, \\
\tilde{Z}(t, 1) = \tilde{Z}(t, 1), & t \geq 0, \\
\tilde{Z}(0, x) = \tilde{Z}_0(x), & x \in [0, 1].
\end{cases}
\end{align*}
\]

(32)

Here \( P(x, y) : \{ (x, y) \mid 0 \leq y \leq x \leq 1 \} \rightarrow \mathbb{R}^{n \times n} \) is a matrix value function to be determined. Noting that the boundary condition \( \tilde{W}(t, 1) = \theta \), similar to (17), we can get that \( P(x, y) \) must satisfy

\[
\begin{align*}
AP_{xx}(x, x) - P_{yy}(x, y) A = P(x, y) B + CP(x, y), \\
AP_x(x, x) + AP_{xx}(x, x) + P_y(x, x) A = -B - C, \\
P(x, x) A - AP(x, x) = O, \\
P(1, y) = O,
\end{align*}
\]

(33)
and the gain matrix \( G(x) \) should be taken as
\[
G(x) = -P(x,0)A. 
\] (34)

**Lemma 6** Let (H) hold. Then the solution to (33) is given by
\[
P(x,y) = -\sum_{n=0}^{\infty} \frac{(1-x) [(1-x)^2 - (1-y)^2]^n [A^{-1}(B+C)]^{n+1}}{n!(n+1)!2^{2n+1}}, 0 \leq y \leq x \leq 1. 
\] (35)

The proof is similar to that of theorem 1, thus we omit it.

Next we seek the inverse transformation of (31), and for any \( \widetilde{Z}(t,x) \) which satisfies (32), define
\[
\tilde{W}(t,x) = \tilde{Z}(t,x) + \int_0^t Q(x,y)\tilde{Z}(t,y)dy. 
\] (36)

Here \( Q(x,y) : \{(x,y)|0 \leq y \leq x \leq 1\} \rightarrow \mathbb{R}^{n \times n} \) is a matrix value function to be determined. In order to make \( \tilde{W}(t,x) \) satisfy (30), it is easy to know that \( Q(x,y) \) should satisfy
\[
\begin{aligned}
AQ_{xx}(x,y) - Q_{yy}(x,y)A &= -BQ(x,y) - Q(x,y)C, \\
AQ'(x,y) + AQ_x(x,y) + Q_y(x,y)A &= -B - C, \\
Q(x,x)A - AQ(x,x) &= O, \\
Q(1,y) &= O.
\end{aligned}
\]

Similar to lemma 6, if (H) holds, then
\[
Q(x,y) = -\sum_{n=0}^{\infty} \frac{(1-x) [(1-y)^2 - (1-x)^2]^n [A^{-1}(B+C)]^{n+1}}{n!(n+1)!2^{2n+1}}, 0 \leq y \leq x \leq 1. 
\]

Meanwhile, for any \( \tilde{Z}_0(x) \in [L^2(0,1)]^n \), a similar result to lemma 4 holds. Furthermore we have the following lemma.

**Lemma 7** Let the gain matrix \( G(x) \) be given by (34). Then for any initial function \( \tilde{W}_0 \in [L^2(0,1)]^n \), the error system (30) exists a unique solution \( \tilde{W}(t,x) \in C(0,\infty;[L^2(0,1)]^n) \). In addition, for some \( \tilde{M} > 0 \),
\[
\|\tilde{W}(t,.)\|_{[L^2(0,1)]^n} \leq \tilde{M}Q(t)\|\tilde{W}_0\|_{[L^2(0,1)]^n}, t \geq 0.
\]

The proof is similar to that of theorem 1, thus we omit it.

Since an approximation of \( W(t,x) \) can be obtained from the observer (29), based on (27), we design the feedback controller as
\[
U(t) = \int_0^1 K(1,y)\tilde{W}(t,y)dy. 
\] (37)

Then the closed-loop system becomes
\[
\begin{aligned}
\frac{C}{\delta} D_t^{\nu(\alpha)} W(t,x) &= AW_{xx}(t,x) + BW(t,x), \ t \geq 0, \ x \in (0,1), \\
W(t,0) &= \theta, t \geq 0, \\
W(t,1) &= \int_0^1 K(1,y)\tilde{W}(t,y)dy, t \geq 0, \\
\frac{C}{\delta} D_t^{\nu(\alpha)} \tilde{W}(t,x) &= \tilde{W}_{xx}(t,x) + B\tilde{W}(t,x) + G(x) [\tilde{W}_x(t,0) - W_x(t,0)], \ t \geq 0, \ x \in (0,1), \\
\tilde{W}(t,0) &= \theta, t \geq 0, \\
\tilde{W}(t,1) &= \int_0^1 K(1,y)\tilde{W}(t,y)dy, t \geq 0, \\
W(0,x) &= W_0(x), \ \tilde{W}(0,x) = \tilde{W}_0(x), \ x \in [0,1].
\end{aligned} 
\] (38)
Theorem 2  For any \(\left( W_0, \tilde{W}_0 \right) \in [H^2(0, 1)]^{2n} \) with the compatibility conditions

\[
W_0(0) = 0, \quad W_0(1) = \int_0^1 K(1, y)\tilde{W}_0(y)dy,
\]

\[
\tilde{W}_0(0) = 0, \quad \tilde{W}_0(1) = \int_0^1 K(1, y)\tilde{W}_0(y)dy,
\]

(39) and

\[
W_0'(0) = 0, \quad W_0'(1) = \int_0^1 K(1, y)\tilde{W}'_0(y)dy,
\]

\[
\tilde{W}_0'(0) = 0, \quad \tilde{W}_0'(1) = \int_0^1 K(1, y)\tilde{W}'_0(y)dy,
\]

(40) and the stability of (41) is equivalent to that of (38). Here \( R \left( \tilde{Z} \right) (t, x) \) is given by

\[
R \left( \tilde{Z} \right) (t, x) = \int_0^t K(x, y) \left( \tilde{Z}(t, y) - \int_0^y P(r, y)\tilde{Z}(t, r)dr \right) dy.
\]

(42)

Let \( Y(t, x) = Z(t, x) - R \left( \tilde{Z} \right) (t, x) \). It is easy to verify that \( Y(t, x) \) satisfies

\[
\begin{cases}
C \cdot D_t^{\nu} Y(t, x) = AY_{xx}(t, x) - CY(t, x) + f(t, x), & t \geq 0, \quad x \in (0, 1), \\
Y(t, 0) = Y(t, 1) = \theta, & t \geq 0,
\end{cases}
\]

(41)

and

\[
f(t, x) = A \left( R(\tilde{Z}) \right)_{xx}(t, x) - CR \left( \tilde{Z} \right) (t, x) - R \left( C \cdot D_t^{\nu} \tilde{Z} \right) (t, x).
\]

(43)

By compatibility conditions (39) and (40), it follows from (33) that

\[
\tilde{Z}_0(0) = \tilde{Z}_0(1) = \tilde{Z}_0'(0) = \tilde{Z}_0'(1) = \theta.
\]

Noting that \( \tilde{Z}_0(x) \in [H^2(0, 1)]^n \) implies \( \tilde{Z}_0'(x), \tilde{Z}_0''(x) \in [L^2(0, 1)]^n \). For the “\( \tilde{Z} \)”-part of system (41), if \( \tilde{Z}_0'(x) \) and \( \tilde{Z}_0''(x) \) are chosen as the initial functions, the corresponding unique solutions \( \tilde{Z}_x(t, x) \) and \( \tilde{Z}_{xx}(t, x) \) can be obtained respectively. By the argument in lemma 4, we obtain, for some \( M_0 > 0 \),

\[
\left\| \tilde{Z}(t, \cdot) \right\|_{[L^2(0, 1)]^n} + \left\| \tilde{Z}_x(t, \cdot) \right\|_{[L^2(0, 1)]^n} + \left\| \tilde{Z}_{xx}(t, \cdot) \right\|_{[L^2(0, 1)]^n}^2 \leq M_0 Q(t) \left\| \tilde{Z}_0 \right\|_{[H^2(0, 1)]^n}^2, \quad t \geq 0.
\]
This, jointly with (42) and (43), implies that
\[ \|f(t,x)\|^2_{L^2([0,1])} \leq MQ(t) \|Z_0\|^2_{H^2([0,1])}, \quad t \geq 0, \]
where \( M > 0 \).

Next, we choose the function
\[ V(t) = \frac{1}{2} \int_0^1 Y^\top(t,x)Y(t,x)dx, \quad t \geq 0. \]

Similar to the proof of lemma 4, we could get
\[ \frac{\partial}{\partial t} D_t^{w(\alpha)} V(t) \leq -\mu V(t) + \int_0^1 Y^\top(t,x)f(t,x)dx. \]

Noting that
\[ \int_0^1 Y^\top(t,x)f(t,x)dx \leq \frac{1}{2} \int_0^1 Y^\top(t,x)Y(t,x)dx + \frac{1}{2} \int_0^1 f^\top(t,x)f(t,x)dx \]
\[ = V(t) + \frac{1}{2} \|f(t,x)\|^2_{L^2([0,1])}, \quad t \geq 0, \]
we have
\[ \frac{\partial}{\partial t} D_t^{w(\alpha)} V(t) \leq -(\mu - 1)V(t) + MQ(t) \|Z_0\|^2_{H^2([0,1])} \Delta - aV(t) + bQ(t), \quad t \geq 0. \]

Hence, there exists a nonnegative function \( P(t) \) satisfying
\[ \frac{\partial}{\partial t} D_t^{w(\alpha)} V(t) + P(t) = -aV(t) + bQ(t), \quad t \geq 0. \quad (44) \]

By taking the Laplace transform on both sides of (44), we can get
\[ \left( \int_0^1 \omega(\alpha)s^{\alpha}d\alpha \right) \tilde{V}(s) - \left( \int_0^1 \omega(\alpha)s^{\alpha-1}d\alpha \right) V(0) + \tilde{P}(s) = -a\tilde{V}(s) + b\tilde{Q}(s), \quad s \in \mathbb{C}, \quad (45) \]
where \( \tilde{V}(s), \tilde{P}(s) \) denote the Laplace transforms of the functions \( V(t) \) and \( P(t) \). It follows from (45) that
\[ \tilde{V}(s) = \frac{V(0)}{s} - V(0) \tilde{F}_2(s) + \left( b\tilde{Q}(s) - \tilde{P}(s) \right) \tilde{F}_2(s), \quad (46) \]
where \( \tilde{F}_2(s) = \frac{1}{\left( \int_0^1 \omega(\alpha)s^{\alpha}d\alpha + a \right)} \). Denoting \( F_2(t) \triangleq \mathcal{L}^{-1}\{\tilde{F}_2(s)\} \), and again, by taking the inverse Laplace transform on both sides of (46), we have
\[ V(t) = V(0) - aV(0) \int_0^t F_2(\tau)d\tau + \int_0^t (bQ(\tau) - P(\tau)) F_2(t-\tau)d\tau \]
\[ \triangleq H(t) V(0) + \int_0^t (bQ(\tau) - P(\tau)) F_2(t-\tau)d\tau, \quad t \geq 0. \]

This, together with the fact that \( P(t) \) is a nonnegative function, leads to
\[ V(t) \leq H(t) V(0) + b \int_0^t Q(\tau) F_2(t-\tau)d\tau, \quad t \geq 0. \quad (47) \]
According to lemma 3, it follows that

$$
\lim_{t \to \infty} H(t) = \lim_{t \to \infty} \left( 1 - a \int_0^t F_2(\tau) d\tau \right) = 0. 
$$

(48)

Since \( \lim_{t \to \infty} Q(t) = 0 \), we know that for any \( \varepsilon > 0 \), there is a \( T > 0 \) such that

$$
Q(t) \leq \varepsilon \text{ for any } t \geq T.
$$

(49)

Define \( c \triangleq \max \{ Q(t) : t \in [0, T] \} \). Here the existence of constant \( c \) is guaranteed by the continuity of \( Q(t) \). By (49), we have

$$
\int_0^t Q(\tau) F_2(t - \tau) d\tau = \int_0^T Q(\tau) F_2(t - \tau) d\tau + \int_T^t Q(\tau) F_2(t - \tau) d\tau \\
\leq c \int_0^T F_2(t - \tau) d\tau + \varepsilon \int_0^{t-T} F_2(\tau) d\tau. 
$$

(50)

According to the final value theorem, we get

$$
\lim_{t \to \infty} F_2(t) = \lim_{s \to 0} \hat{F}_2(s) = \lim_{s \to 0} \frac{s}{\int_0^t w(\alpha) s^a d\alpha + a} = 0,
$$

namely, for the above \( \varepsilon > 0 \), there exists \( T' > T \) such that

$$
F_2(t - \tau) \leq \varepsilon \text{ for any } t \geq T'.
$$

(51)

It follows from (50) and (51) that

$$
\int_0^t Q(\tau) F_2(t - \tau) d\tau \leq cT'\varepsilon + \frac{\varepsilon}{a}.
$$

The arbitrariness of \( \varepsilon \) implies that

$$
\lim_{t \to \infty} \int_0^t Q(\tau) F_2(t - \tau) d\tau = 0. 
$$

(52)

Based on (48) and (52), we have \( \lim_{t \to \infty} V(t) = 0 \), which leads to

$$
\lim_{t \to \infty} \| Y(t, \cdot) \|_{L^2(0,1)}^n = 0.
$$

Noting that \( Y(t, x) = Z(t, x) - R\left( \hat{Z} \right)(t, x) \), we have

$$
\lim_{t \to \infty} \| Z(t, \cdot) \|_{L^2(0,1)}^n = 0,
$$

that is, system (41) is asymptotically stable. Equivalently, system (38) is asymptotically stable.

**Remark 2** In theorem 2, we prove the asymptotical stability of \( \left( W(t, \cdot), \hat{W}(t, \cdot) \right) \) in \( L^2(0,1) \) under the assumption that \( W_0, \hat{W}_0 \in H^2(0,1) \), which means that the initial function is smoother than the solution to system (1).
5 Output feedback for collocated control

In this section, we consider \( W_x(t, 1) \) as the output measurement which is collocated with the input, and \( W(t, x) \) is still unmeasurable. Note that the argument is similar to that in section 4, we omit most of the proof details. In this case, we propose an output observer \( \hat{W}(t, x) \) for system (1) with output \( W_x(t, 1) \) as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
C D_t^\alpha \hat{W}(t, x) = A W_{xx}(t, x) + B \hat{W}(t, x) + G(x) \left[ \hat{W}(t, 1) - W_x(t, 1) \right], \ t \geq 0, \ x \in (0, 1), \\
\hat{W}(0, x) = 0, \ t \geq 0, \\
\hat{W}(t, 0) = \theta, \ t \geq 0, \\
\hat{W}(0, x) = \hat{W}_0(x), \ x \in [0, 1].
\end{array} \right.
\]
\]

(53)

And the observer error \( \hat{W}(t, x) \) is governed by

\[
\begin{align*}
\left\{ \begin{array}{l}
C D_t^\nu \hat{W}(t, x) = A \hat{W}_{xx}(t, x) + B \hat{W}(t, x) + G(x) \hat{W}(t, 1), \ t \geq 0, \ x \in (0, 1), \\
\hat{W}(t, 0) = \theta, \ t \geq 0, \\
\hat{W}(t, 1) = \theta, \ t \geq 0, \\
\hat{W}(0, x) = \hat{W}_0(x) - W_0(x), \ x \in [0, 1].
\end{array} \right.
\]

(54)

Here, we introduce the backstepping transformation: for any \( \hat{W}(t, x) \) satisfying (54), assume that

\[
\tilde{Z}(t, x) = \hat{W}(t, x) - \int_x^1 R(x, y) \hat{W}(t, y) dy,
\]

(55)

to convert (54) into

\[
\begin{align*}
\left\{ \begin{array}{l}
C D_t^\nu \tilde{Z}(t, x) = A \tilde{Z}_{xx}(t, x) - C \tilde{Z}(t, x), \ t \geq 0, \ x \in (0, 1), \\
\tilde{Z}(t, 0) = \theta, \ \tilde{Z}(t, 1) = \theta, \ t \geq 0, \\
\tilde{Z}(0, x) = \tilde{Z}_0(x), \ x \in [0, 1].
\end{array} \right.
\]

(56)

Then, we could get

\[
G(x) = R(x, 1) A,
\]

(57)

and

\[
R(x, y) = - \sum_{n=0}^{\infty} \frac{x (x^2 - y^2)^n}{n! (n + 1)! 2^{2n+1}} \left[ A^{-1}(B + C) \right]^{n+1}, \ 0 \leq y \leq x \leq 1.
\]

Once again, we seek the inverse transformation of (55), for any \( \tilde{Z}(t, x) \) which satisfies (54), define

\[
\tilde{W}(t, x) = \tilde{Z}(t, x) + \int_x^1 S(x, y) \tilde{Z}(t, y) dy,
\]

and can conclude that \( S(x, y) \) is of the form

\[
S(x, y) = - \sum_{n=0}^{\infty} \frac{x (y^2 - x^2)^n}{n! (n + 1)! 2^{2n+1}} \left[ A^{-1}(B + C) \right]^{n+1}, \ 0 \leq y \leq x \leq 1.
\]
Lemma 8 Let the gain matrix $G(x)$ be given by (57). Then for any initial function $\bar{W}_0 \in [L^2(0, 1)]^n$, (54) exists a unique solution $\bar{W}(t, x) \in \mathcal{C} \left(0, \infty; [L^2(0, 1)]^n\right)$. In addition, for some $\bar{M} > 0$,

$$\left\|\bar{W}(t, \cdot)\right\|_{[L^2(0, 1)]^n}^2 \leq \bar{M}Q(t) \left\|\bar{W}_0\right\|_{[L^2(0, 1)]^n}^2, \ t \geq 0.$$

Based on the output observer (53), the output feedback controller (37) is designed as

$$U(t) = \int_0^1 K(1, y)\bar{W}(t, y)dy,$$

with the corresponding closed-loop system:

$$\begin{align*}
\frac{d}{dt}W(t, x) &= AW_x(t, x) + BW(t, x), \ t \geq 0, \ x \in (0, 1), \\
W(t, 0) &= \theta, \ t \geq 0, \\
W(t, 1) &= \int_0^1 K(1, y)\bar{W}(t, y)dy, \ t \geq 0, \\
\frac{d}{dt}\bar{W}(t, x) &= A\bar{W}_x(t, x) + B\bar{W}(t, x) + G(x) \left[\bar{W}(t, 1) - W(t, 1)\right], \ t \geq 0, \ x \in (0, 1), \\
\bar{W}(t, 0) &= \theta, \ t \geq 0, \\
\bar{W}(t, 1) &= \int_0^1 K(1, y)\bar{W}(t, y)dy, \ t \geq 0, \\
W(0, x) &= W_0(x), \ \bar{W}(0, x) = \bar{W}_0(x), \ x \in [0, 1].
\end{align*}$$

Theorem 3 For any initial function $\left(W_0, \bar{W}_0\right) \in \left[H^2(0, 1)\right]^{2n}$ with the compatibility conditions

$$\begin{align*}
W_0(0) &= \theta, \ W_0(1) = \int_0^1 K(1, y)\bar{W}_0(y)dy, \\
\bar{W}_0(0) &= \theta, \ \bar{W}_0(1) = \int_0^1 K(1, y)\bar{W}_0(y)dy,
\end{align*}$$

and

$$\begin{align*}
W_0'(0) &= \theta, \ W_0'(1) = \int_0^1 K(1, y)\bar{W}_0'(y)dy, \\
\bar{W}_0'(0) &= \theta, \ \bar{W}_0'(1) = \int_0^1 K(1, y)\bar{W}_0'(y)dy,
\end{align*}$$

(58) has a unique solution on $[L^2(0, 1)]^{2n}$. Moreover, (58) is asymptotically stable, i.e.,

$$\lim_{t \to \infty} \left\|(W(t, \cdot), \bar{W}(t, \cdot))\right\|_{[L^2(0, 1)]^{2n}} = 0.$$

6 An example

In this section, we give an example to illustrate our results.

In the system (1), we take $w(\alpha) = 2\alpha, \ \alpha \in (0, 1)$, the coefficient matrices

$$A = \begin{pmatrix} 4 & 1 \\ 0 & 1/4 \end{pmatrix}, \ B = \begin{pmatrix} 1/4 & -1 \\ 0 & 4 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
and the initial functions \( W_0 (x) = (x, x)^	op, \hat{W}_0 (x) = (x, 2x)^	op \). We can verify that these coefficient matrices satisfy (H) by direct calculation.

Let \( \sigma \left( A \frac{\partial^2}{\partial x^2} + B \right) \) be the spectrum set of operator \( A \frac{\partial^2}{\partial x^2} + B \) with certain boundary conditions.

If \( U (t) = 0 \), a direct computation shows that

\[
\sigma \left( A \frac{\partial^2}{\partial x^2} + B \right) = \left\{ \lambda_{1,k}, \lambda_{1,k} = \frac{1}{4} - 4k^2 \pi^2, k \in \mathbb{N} \right\} \cup \left\{ \lambda_{2,k}, \lambda_{2,k} = 4 - \frac{1}{4} k^2 \pi^2, k \in \mathbb{N} \right\}.
\]

Then \( \lambda_{2,1} = 4 - \frac{\pi^2}{4} > 0 \). Thus, the system (1) is unstable.

Next, we consider the non-collocated output \( Y_{\text{out}} (t) = W_x (t, 0) \). The unique solution of the system (1) is denoted by \( W (t, x) = (W_1 (t, x), W_2 (t, x))^	op \), where \( W_1 (t, x) \) and \( W_2 (t, x) \) are real functions. With these parameters, the matrix value function \( K(x, y) \) in (18) is as follows:

\[
K (x, y) = -\sum_{n=0}^{\infty} \frac{y (x^2 - y^2)^n}{n! (n + 1)! 2^{2n+1}} \begin{pmatrix} \frac{5}{16} n+1 \\ 0 \\ \frac{5}{16} 12^{10^{n+1}} \end{pmatrix}.
\]

Hence,

\[
K (1, y) = -\sum_{n=0}^{\infty} \frac{y (1 - y^2)^n}{n! (n + 1)! 2^{2n+1}} \begin{pmatrix} \frac{5}{16} n+1 \\ 0 \\ \frac{5}{16} 12^{10^{n+1}} \end{pmatrix}.
\]

From lemma 6 and (34), we get

\[
G (x) = \sum_{n=0}^{\infty} \frac{(1 - x) [1 - (1 - x)^2]^n}{n! (n + 1)! 2^{2n+1}} \begin{pmatrix} 4 (\frac{5}{16}) n+1 \\ 0 \\ \frac{5}{16} 12^{10^{n+1}} \end{pmatrix}.
\]

The output feedback controller \( U (t) \) is given by

\[
U (t) = \int_0^1 K (1, y) \hat{W} (t, y) dy.
\]

The closed-loop system of (1) is as follows:

\[
\begin{cases}
\tilde{D} \tilde{D} \tilde{W} (t, x) = AW_{xx} (t, x) + BW (t, x), & t \geq 0, \ x \in (0, 1), \\
W (t, 0) = \theta, & t \geq 0, \\
W (t, 1) = \int_0^1 K (1, y) \hat{W} (t, y) dy, & t \geq 0,
\end{cases}
\]

\[
\begin{cases}
\tilde{D} \tilde{D} \tilde{W} (t, x) = A \hat{W}_{xx} (t, x) + B \hat{W} (t, x) + G (x) \left[ \hat{W}_x (t, 0) - W_x (t, 0) \right], & t \geq 0, \ x \in (0, 1), \\
\hat{W} (t, 0) = \theta, & t \geq 0, \\
\hat{W} (t, 1) = \int_0^1 K (1, y) \hat{W} (t, y) dy, & t \geq 0,
\end{cases}
\]

\[
\begin{cases}
W (t, x) = W_0 (x), & \hat{W} (0, x) = \hat{W}_0 (x), \ x \in [0, 1] .
\end{cases}
\]

Then, from theorem 2, we get that the system (1) is asymptotically stable.
7 Conclusion

In this paper, we design the feedback controllers to realize the boundary feedback stabilization for the distributed-order fractional reaction diffusion systems. We introduce a target system and analyze its asymptotical stability by using the Poincaré inequality and differential inequality. For the first time, a new state feedback control law which can realize the asymptotical stability of the DOFSs (1) is designed by the backstepping transformation, and an explicit kernel matrix function of the transformation is found. Based on these, we design the corresponding observers and feedback controllers for the non-collocated output and the collocated output, respectively, to realize the asymptotical stability of the studied system. And we provide an example to illustrate our results. In the future work, it seems to be an interesting problem to study the stabilization of DOFSs with time-varying delay by using the backstepping method and differential inequality.

References:


