Bifurcations of traveling wave solutions of a (3+1)-dimensional nonlinear model generated by the Jaulent-Miodek hierarchy

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Abstract: We study bifurcation of traveling wave solutions of a class of (3+1)-dimensional nonlinear evolution equations generated by the Jaulent-Miodek hierarchy. We obtain phase portraits of the nonlinear transformation system according to the different bifurcation regions of parameters. Different kinds of traveling wave solutions, such as the periodic wave solutions, solitary wave solutions, kink wave solutions and anti-kink wave solutions are found to exist under certain parameter conditions, and the exact solutions of traveling waves are obtained.

Key words: (3+1)-dimensional nonlinear evolution equation; bifurcation; traveling wave solution; exact solution

2010 MSC: 34A34, 37H20

ビフリックの進行波解方程式の構造

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Received date: 2017-09-01
Foundation item: The Natural Science Foundation of China (11772007, 11372014, 11072007, 11290152, 11072008); Beijing Natural Science Foundation (1172002, 1122001)

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1 Introduction

There are many important nonlinear problems in physics, chemistry, biology, engineering technology etc. Many nonlinear phenomena are modeled by nonlinear partial differential equations. The investigation of the traveling wave solutions to nonlinear evolution equations plays an important role in the mathematical physics.

To find exact traveling wave solutions for a given nonlinear wave system, a lot of methods have been developed since 1970's, such as elliptic integral method\cite{1}, Hirota bilinear method\cite{2}, extended tanh-function method\cite{3} and homogeneous balance method\cite{4}. In 1993, Camassa and Holm\cite{5} used Hamiltonian methods to derive a new completely integrable dispersive shallow water wave equation, and they found that there exists a non-smooth peaked solitary wave solution(so called peakon). In 2007, Abdou and Elhanbaly\cite{6} used extended Jacobi elliptic function expansion method to construct the exact periodic solutions of some polynomials or nonlinear evolution equations, and obtained many exact traveling wave solutions including the new solitary and shock wave solution. From 2007 to 2009, Li\cite{7-8} obtained the existence of solitary wave, kink wave and periodic wave solutions of a class of singular reaction-diffusion equations by using some effective methods from the dynamical systems theory. In 2014, Huang and Lin\cite{9} established the existence of nontrivial traveling wave solutions by constructing upper and lower solutions and proved the nonexistence of traveling wave solutions by applying the theory of asymptotic spreading. In 2015, Fan and Li\cite{10} investigated the bifurcations of traveling wave solutions of a generalized Dullin-Gottwald-Holm equation by using the method of planar dynamical systems, and they found different kinds of traveling wave solutions including the solitary wave solution, the peakon wave solution and the periodic cusp wave solution. In 2016, Li et al\cite{11} used the method of dynamical systems, and gave bifurcation diagrams and explicit exact parametric representations of the solutions of the shallow water equations, including solitary wave solution, periodic wave solution, peakon solution, periodic peakon solution and compacton solution under different parameter conditions. In 2017, Li et al\cite{12} obtained the existence of solutions, including three forms of solitary wave solutions and four exact bounded periodic wave solutions under different conditions.

Jaulent-Miodek equation is one of the important nonlinear dynamical models in physics. The nonlinear differential equation of this special type has no uniform method. In 2009, Wazwaz\cite{13} investigated four (2+1)-dimensional nonlinear evolution equations generated by the Jaulent-Miodek hierarchy and obtained multiple singular kink solutions. In 2012, Wazwaz\cite{14} studied four (3+1)-dimensional nonlinear evolution equations generated by the Jaulent-Miodek hierarchy and derived multiple soliton solutions for each equation by using a simplified form of the Hirota's method. In 2016, Sahoo and Saha\cite{15} investigated a class of time-fractional coupled Jaulent-Miodek equation by tanh method and \((G'/G)\)-expansion method by means of fractional complex transform and acquired a new exact analytical solutions of Jaulent-Miodek. In 2017, Li et al\cite{16} obtained the solitary wave, kink(anti-kink) wave and periodic wave solutions for the first equation of a class of (3+1)-dimensional nonlinear equation by some effective methods from the dynamical systems theory.

In this paper, we study a class of (3+1)-dimensional nonlinear evolution equations generated by the Jaulent-
Miodek hierarchy. We derive periodic wave, solitary wave, kink wave and anti-kink wave solutions of these equations when the values of parameters vary. Firstly, we transform the Jaulent-Miodek model into a system of ordinary differential equations and analyze the bifurcation of equilibrium points of the system. Secondly, according to the different values of bifurcation parameter, we obtain phase portraits and profiles of waves of the system. Finally, we obtain exact traveling wave solutions of the Jaulent-Miodek model.

2 Nonlinear evolution equations generated by the Jaulent-Miodek hierarchy and its equilibrium bifurcation

In this paper we will study the (3+1)-dimensional nonlinear model generated by the Jaulent-Miodek hierarchy\(^{[14]}\). The (3+1)-dimensional model is developed in the form

\[
w_t = 2(w_{xx} - 2w^3)_x - \frac{3}{4}(\partial_x^{-1} w_{yy} - 2w_x \partial_x^{-1} w_y - 6ww_y) - \frac{3}{4} \partial_x^{-1} w_{zz} - \frac{1}{4} w_z - \frac{1}{2} w_y,
\]

where the operator \(\partial_x^{-1}\) is the inverse of \(\partial_x\) with \(\partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x = id\) (identity operator), and

\[
(\partial_x^{-1} f)(x) = \int_{-\infty}^{x} f(t) dt,
\]

under the decaying condition at infinity. To remove the integral term in Eq. (1) we use the potential

\[
w(x, y, z, t) = u_x(x, y, z, t),
\]

to carry Eq. (1) to the equation

\[
u_{xt} - 2u_{xxxx} + 12u_x^2 u_{xx} + \frac{3}{4} u_{yy} - \frac{3}{2} u_{xx} u_y - \frac{9}{2} u_x u_{xy} + \frac{3}{4} u_{zz} + \frac{1}{4} u_{xz} + \frac{1}{2} u_{xy} = 0.
\]

To find traveling wave solutions of Eq. (1), we use the wave transformation \(\xi = ax + by + cz - dt\), where \(d\) is the propagating wave velocity. According to the physical significance of traveling wave solutions of Eq. (1), let \(d > 0, a^2 + b^2 + c^2 \neq 0\). Now, substituting \(u(x, y, z, t) = u(ax + by + cz - dt) = u(\xi)\) into Eq. (4), we have the traveling wave equation

\[
-2a^4 u^{(4)} + 12a^4 (u')^2 u'' - 6a^2 bu'u'' + pu'' = 0,
\]

where \(p = -ad + \frac{3}{4} b^2 + \frac{3}{4} c^2 + \frac{1}{4} ac + \frac{1}{2} ab\).

Integrating Eq. (5) with respect to \(\xi\) once, we have

\[
-2a^4 u''' + 4a^4 (u')^3 - 3a^2 b(u')^2 + pu' + C_0 = 0.
\]

Let \(u' = \phi, C_0 = 0\), Eq. (6) becomes

\[
-2a^4 \phi'' + 4a^4 \phi^3 - 3a^2 b\phi^2 + p\phi = 0.
\]

With the function \(\phi' = \eta\), Eq. (7) becomes a system of ordinary differential equation, as follows:

\[
\frac{d\phi}{d\xi} = \eta,
\]

\[
\frac{d\eta}{d\xi} = 2\phi^3 - 2m\phi^2 + 2n\phi,
\]

where \(m = \frac{3b}{4a^2}, n = \frac{p}{4a^4}\).
Obviously, Eq. (8) is a Hamiltonian system with Hamiltonian function

\[
H(\phi, \eta) = \frac{1}{2} \phi^2 - n\phi^2 + \frac{2}{3} \phi^3 - \frac{1}{2} \phi^4. \tag{9}
\]

For a fixed \( h \), the level curve \( H(\phi, \eta) = h \) defined by Eq. (9) determines a set of solution curves of system (8), which includes different branches of curves. To study the phase portraits of system (8), let

\[
f(\phi) = 2\phi^3 - 2m\phi^2 + 2n\phi = 2\phi(\phi^2 - m\phi + n), \tag{10}
\]

\( \Delta_1 = m^2 - 4n \), then \( \phi_1 = \frac{m + \sqrt{\Delta_1}}{2} \) and \( \phi_2 = \frac{m - \sqrt{\Delta_1}}{2} \) are the roots of \( f(\phi) = 0 \). The equilibrium points of system (8) are given as follows:

1) When \( \Delta_1 < 0 \), system (8) has only one trivial equilibrium point \( (0, 0) \);
2) When \( \Delta_1 > 0, n \neq 0 \), system (8) has three equilibrium points \((0, 0), (\phi_1, 0) \) and \((\phi_2, 0) \); when \( \Delta_1 > 0, n = 0 \), system (8) has a second-order equilibrium point \((0, 0) \) and an equilibrium point \((m, 0) \);
3) When \( \Delta_1 = 0, n \neq 0 \), system (8) has a second-order equilibrium point \((\phi_1, 0) = (\phi_2, 0) \) and an equilibrium point \((0, 0) \); when \( \Delta_1 = 0, n = 0 \), system (8) has a third-order equilibrium point \((\phi_1, 0) = (\phi_2, 0) = (0, 0) \).

The coefficient matrix of the linearized system (8) at the equilibrium points is

\[
M_\phi = M(\phi, 0) = \begin{pmatrix}
0 & 1 \\
\phi' & 0
\end{pmatrix}. \tag{11}
\]

The Jacobian of the linearized system (8) is

\[
J(\phi, \eta) = \begin{vmatrix}
0 & 1 \\
\phi' & 0
\end{vmatrix} = -\phi'(\phi), \tag{12}
\]

the Jacobian of the linearized system (8) at the equilibrium points is given by

\[
J(\phi, 0) = -\phi'(\phi) = -6\phi_i^2 + 4m\phi_i - 2n, \tag{13}
\]

where \( i = 0, 1, 2 \), and \( \phi_0 = 0 \).

Substituting the equilibrium points of system (8) into Eq. (13), we have

\[
J_0 = J(0, 0) = -2n, \quad J_1 = J(\phi_1, 0) = (-\sqrt{m^2 - 4n - m})\sqrt{m^2 - 4n},
\]

\[
J_2 = J(\phi_2, 0) = (-\sqrt{m^2 - 4n + m})\sqrt{m^2 - 4n},
\]

by the qualitative theory of planar differential systems, for the equilibrium points of system (8), if \( J_i < 0 \), then the equilibrium point \((\phi_i, 0) \) is a saddle point; if \( J_i > 0 \) and \( \text{trace } M_i > 4J_i < 0 \), then the equilibrium point \((\phi_i, 0) \) is a center point; if \( J_i = 0, m^2 + n^2 = 0 \), then the equilibrium point \((\phi_i, 0) \) is a degenerate saddle point; if \( J_i = 0, m^2 + n^2 \neq 0 \), then the equilibrium point \((\phi_i, 0) \) is a cusp point.

The discriminant \( \Delta_2 = 2m^2 - 9n \) is determined by the level curve \( H(\phi, 0) = H(0, 0) \). The discriminant \( \Delta_2 \) can determine the number of intersection points of the level curve \( H(\phi, 0) = H(0, 0) \) and \( \phi \)-axis. According to \( \Delta_1 \) and \( \Delta_2 \), the parameters \( m \) and \( n \) can be classified into 15 sets (see Figure 1) so that we can determine the type of the equilibrium points of system (8), where sets

\[
I = \{(m, n) | n > 0, m > \sqrt{\frac{9n}{2}} \}, L_1 = \{(m, n) | n > 0, m = \sqrt{\frac{9n}{2}} \},
\]

\[
II = \{(m, n) | n > 0, 2\sqrt{n} < m < \sqrt{\frac{9n}{2}} \}, L_2 = \{(m, n) | n > 0, m = 2\sqrt{n} \},
\]
The properties of the equilibrium points of system (8) vary with the change of the values of parameters \(m\) and \(n\). We get the following two theorems.

**Theorem 1** If the parameter values \(m\) and \(n\) are in the 15 different sets above respectively, then the types of equilibrium points of system (8) are shown in Table 1, where C, S, CP, DS and NE denote center point, saddle point, cusp point, degenerate saddle point and non-existent, respectively.

<table>
<thead>
<tr>
<th>Set</th>
<th>(I)</th>
<th>(II)</th>
<th>(III)</th>
<th>(IV)</th>
<th>(V)</th>
<th>(VI)</th>
<th>(VII)</th>
<th>(L_1)</th>
<th>(L_2)</th>
<th>(L_3)</th>
<th>(L_4)</th>
<th>(L_5)</th>
<th>(L_6)</th>
<th>(L_7)</th>
<th>(O)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0))</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>CP</td>
<td>C</td>
<td>CP</td>
<td>DS</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((\phi_1,0))</td>
<td>S</td>
<td>S</td>
<td>NE</td>
<td>C</td>
<td>C</td>
<td>S</td>
<td>S</td>
<td>CP</td>
<td>CP</td>
<td>C</td>
<td>CP</td>
<td>S</td>
<td>S</td>
<td>DS</td>
<td></td>
</tr>
<tr>
<td>((\phi_2,0))</td>
<td>C</td>
<td>C</td>
<td>NE</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>C</td>
<td>CP</td>
<td>CP</td>
<td>S</td>
<td>S</td>
<td>CP</td>
<td>DS</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof** We substitute different sets of parameters \(m\) and \(n\) into Eq.(8), and use the qualitative theory of dynamical system and the conclusion above. The theorem can be proved.

**Theorem 2** If the parameter \(m\) and \(n\) are in the 15 different sets above respectively, then the bifurcations of the phase portraits of system (8) can be obtained, which are shown in Figure 2.

**Proof** We take fixed parameter values \(m\) and \(n\) from the 15 sets \(I-O\), these equilibrium points in the 15 sets can be written as \((\phi_{i,j},0)\), and the Hamiltonian values at these equilibrium points are given by \(h_{i,j} = H(\phi_{i,j},0)\), where \(\phi_{0,j} = 0, i = 0, 1, 2; j = 1, \ldots, 15\). By analyzing the Hamiltonian values at equilibrium in different sets of Theorem 1, and using the properties of Hamiltonian system and Maple soft, we choose the proper Hamiltonian values to draw phase portraits of system (8) in the 15 sets. The theorem is proved.

3 Exact traveling wave solutions of system (8) and system (1)

In this section, we will obtain some exact expressions of traveling wave solutions of system (8) and system...
(a) \((m, n) \in I\)  
(b) \((m, n) \in L_1\)  
(c) \((m, n) \in II\)

(d) \((m, n) \in L_2\)  
(e) \((m, n) \in III\)  
(f) \((m, n) \in L_3\)

(g) \((m, n) \in IV\)  
(h) \((m, n) \in L_4\)  
(i) \((m, n) \in V\)

(j) \((m, n) \in L_5\)  
(k) \((m, n) \in VI\)  
(l) \((m, n) \in L_6\)

(m) \((m, n) \in VII\)  
(n) \((m, n) \in L_7\)  
(o) \((m, n) \in O\)

Figure 2  Bifurcations of phase portraits of system (8)
To analyze phase portraits shown in Figure 2, we will primarily analyze periodic orbits, homoclinic orbits and heteroclinic orbits. The following method is also applicable to the other orbits.

### 3.1 Periodic wave solutions of system (8) and system (1)

When \((m, n) \in I\), there are a lot of periodic orbits. Corresponding to the periodic orbit and two special orbits shown in Figure 3(a) defined by \(H(\phi, \eta) = h\), where \(h \in (h_{2.1}, h_{0.1})\), we see the level curve intersects the axis \(\phi\) at \(\varphi_1, \varphi_2, \varphi_3\) and \(\varphi_4\), with \(\varphi_1 > \varphi_2 > \varphi_3 > \varphi_4\), and the expression of the orbits of system (8) is given by

\[
\eta = \pm \sqrt{\phi^4 - \frac{4}{3} m\phi^3 + 2n\phi^2 + 2h_1} = \pm \sqrt{(\varphi_1 - \phi)(\varphi_2 - \phi)(\varphi - \varphi_3)(\varphi - \varphi_4)}. \tag{14}
\]

Substituting (14) into \(\frac{d\phi}{d\xi} = \eta\) and integrating them along the orbit by using elliptic integral method, and completing the integral, we obtain one of the periodic traveling wave solutions

\[
\phi_1(\xi) = \frac{(\varphi_2 - \varphi_3)\varphi_4\text{sn}^2\left(\frac{\xi}{g}, k\right) - (\varphi_2 - \varphi_4)\varphi_3}{(\varphi_2 - \varphi_3)\text{sn}^2\left(\frac{\xi}{g}, k\right) - (\varphi_2 - \varphi_4)}, \tag{15}
\]

where \(g = \frac{2}{\sqrt{(\varphi_1 - \varphi_3)(\varphi_2 - \varphi_4)}}\), \(k = \frac{\sqrt{(\varphi_2 - \varphi_3)(\varphi_1 - \varphi_4)}}{(\varphi_1 - \varphi_3)(\varphi_2 - \varphi_4)}\).

From \(u_1(\xi) = \int \phi_1(\xi) d\xi\), we obtain \(u_1(x, y, z, t) = \frac{\partial u_1(x, y, z, t)}{\partial x}\) by integrating (15) and also obtain periodic wave solution \(u_1 = a\phi_1\) of system (1) by using the potential (3)(see Figure 3).

![Figure 3 Periodic orbit and periodic wave solutions](image)

### 3.2 Solitary wave solutions of valley type of system (8) and system (1)

When \((m, n) \in VI\), there is a homoclinic orbit. Corresponding to the homoclinic orbit and a special orbit shown in Figure 4(a) defined by \(H(\phi, \eta) = h_{1.1}\), we see the level curve intersects the axis \(\phi\) at \(\varphi_5, \varphi_6, \varphi_7\), with \(\varphi_5 > \varphi_6 > \varphi_7\), and the expression of the orbits of system (8) is given by

\[
\eta = \pm \sqrt{(\varphi_5 - \phi)^2(\phi - \varphi_6)(\phi - \varphi_7)}. \tag{16}
\]

Substituting (16) into \(\frac{d\phi}{d\xi} = \eta\) and integrating them along the curve, completing the integral, we obtain one of the solitary traveling wave solutions

\[
\phi_2(\xi) = \varphi_5 - \frac{2(\varphi_5 - \varphi_6)(\varphi_5 - \varphi_7)}{(\varphi_6 - \varphi_7)\cosh\left(\sqrt{(\varphi_5 - \varphi_6)(\varphi_5 - \varphi_7)}\xi\right) + (2\varphi_5 - \varphi_6 - \varphi_7)}. \tag{17}
\]

From \(u_2(\xi) = \int \phi_2(\xi) d\xi\), we obtain \(u_2(x, y, z, t) = \frac{\partial u_2(x, y, z, t)}{\partial x}\) by integrating (17), and also obtain solitary wave solution \(u_2 = a\phi_2\) of valley type of system (1) by using the potential (3)(see Figure 4).
3.3 Solitary wave solutions of peak type of system (8) and system (1)

When \((m, n) \in V_{II}\), there is a homoclinic orbit. Corresponding to the homoclinic orbit and a special orbit shown in Figure 5(a) defined by \(H(\phi, \eta) = h_{2,13}\), we see the level curve intersects the axis \(\varphi_8, \varphi_9\) and \(\varphi_{10}\), with \(\varphi_8 > \varphi_9 > \varphi_{10}\), and the expression of the orbits of system (8) is given by

\[
\eta = \pm \sqrt{(\varphi_{10} - \phi)^2(\phi - \varphi_9)(\phi - \varphi_8)},
\]

Substituting (18) into \(\frac{d\phi}{d\xi} = \eta\) and integrating them along the red upper curve, it follows that

\[
\pm \int_{\phi_0}^{\phi_1} \frac{d\phi}{\sqrt{(\varphi_{11} - \phi)^2(\phi - \varphi_{12})^2}} = \int_{0}^{\xi} ds,
\]

From \(u_3(\xi) = \int \phi_3(\xi)d\xi\), we get \(w_3(x, y, z, t) = \frac{\partial u_3(x, y, z, t)}{\partial x}\) by integrating (19), and also get solitary wave solution \(w_3 = a\phi_3\) of peak type of system (1) by using the potential (c)(see Figure 5).

3.4 Kink (anti-kink) wave solutions of system (8) and system (1)

When \((m, n) \in L_6\), there are two heteroclinic orbits. Corresponding to the heteroclinic orbits shown in Figure 6(a) defined by \(H(\phi, \eta) = h_{1,12}\), we see the level curve intersects the axis \(\phi\) at \(\varphi_{11}\) and \(\varphi_{12}\), with \(\varphi_{11} > \varphi_{12}\), and the expression of the orbits of system (8) is given by

\[
\eta = \pm \sqrt{(\varphi_{11} - \phi)^2(\phi - \varphi_{12})^2},
\]

Substituting (20) into \(\frac{d\phi}{d\xi} = \eta\) and integrating them along the red upper curve, it follows that

\[
\pm \int_{\phi_0}^{\phi_1} \frac{d\phi}{\sqrt{(\varphi_{11} - \phi)^2(\phi - \varphi_{12})^2}} = \int_{0}^{\xi} ds,
\]
where $u_0 = \frac{\varphi_{11} + \varphi_{12}}{2}$. Completing the above integral, we obtain one of the kink traveling wave solutions

$$
\phi_4(\xi) = \frac{\varphi_{11} + \varphi_{12}}{2} + \frac{\varphi_{11} - \varphi_{12}}{2} \tanh\left(\frac{\xi}{4}\right).
$$

(22)

From $u_4(\xi) = \int \phi_4(\xi) d\xi$, $w_4(x, y, z, t) = \frac{\partial u_4(x, y, z, t)}{\partial x}$, we get $u_4(\xi)$ by integrating (22) and also get kink wave solution $w_4 = a\phi_4$ of system (1) by using the potential (c)(see Figure 6).

Substituting (20) into $\frac{d\phi}{d\xi} = \eta$ and integrating them along the blue lower curve, and completing the integral, we obtain one of the anti-kink traveling wave solutions

$$
\phi_4(\xi) = \frac{\varphi_{11} + \varphi_{12}}{2} + \frac{\varphi_{11} - \varphi_{12}}{2} \tanh\left(-\frac{\xi}{4}\right).
$$

(23)

From $u_5(\xi) = \int \phi_5(\xi) d\xi$, $w_5(x, y, z, t) = \frac{\partial u_5(x, y, z, t)}{\partial x}$, we get $u_5(\xi)$ by integrating (21), and also get anti-kink wave solution $w_5 = a\phi_5$ of system (1) by using the potential (c)(see Figure 6).

Through the above analysis, we arrive at the following theorem.

**Theorem 3** If system (8) has a periodic orbit, a homoclinic orbit, or a heteroclinic orbit, then system (1) has a periodic wave solution, a solitary wave solution, or a kink (an anti-kink) wave solution, respectively.

4 Conclusions

In this paper, applying the theory of dynamical systems to the (3+1)-dimensional nonlinear model generated by the Jaulent-Miodek hierarchy, we obtain periodic wave solutions, solitary wave solutions, kink wave solutions and anti-kink wave solutions of the model in the case of specific parameter values. The method is also applicable to the other equations in [14]. In [14], Wazwaz uses the Hereman-Nuseir form to get a simplified form of the Hirota's method, and obtain the expression of soliton solutions of system (1). In this paper, by using the bifurcation theory
of dynamical system, we not only get the exact expression of solitary wave solutions of system (1), but also get the exact expression of periodic wave solutions, kink wave and anti-kink wave solutions of system (1).

5 Acknowledgements

The research project is supported by National Natural Science Foundation of China (11772007, 11372014, 11072007, 11290152 and 11072008) and also supported by Beijing Natural Science Foundation (1172002, 1122001), the International Science and Technology Cooperation Program of China (2014DFR61080), the Funding Project for Academic Human Resources Development in Institutions of Higher Learning under the Jurisdiction of Beijing Municipality(PHRILB), Beijing Key Laboratory on Nonlinear Vibrations and Strength of Mechanical Structures of Mechanical Engineering College of Beijing University of Technology. All authors wish to thank Professor Li Jibin for many valuable suggestions leading to an improvement of this paper.

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